A PRIORI BOUNDS AND MULTIPLE SOLUTIONS FOR SUPERLINEAR INDEFINITE ELLIPTIC PROBLEMS

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Abstract. In this work we study existence and multiplicity questions for positive solutions of second order semilinear elliptic boundary value problems, where the nonlinearity is multiplied by a weight function which is allowed to change sign and vanish on sets of positive measure. We do not impose a variational structure so that techniques from the calculus of variations are not applicable. Under various qualitative assumptions on the nonlinearity we establish a priori bounds and employ bifurcation and fixed point index theory to prove existence and multiplicity results for positive solutions. In an appendix we derive interior L_p-estimates for general elliptic systems of arbitrary order under minimal smoothness hypotheses. Special instances of these results are used in the derivation of a priori bounds.

1 Introduction. In this paper we analyze existence and multiplicity questions for positive solutions of

$$\begin{aligned} \mathcal{A}u &= \lambda u + a(x)f(x,u)u \qquad && \text{in } \Omega\ , \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega\ , \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ of class C^2, that is, $\overline{\Omega}$ is an n-dimensional compact connected C^2-submanifold of $\mathbb{R}^n$ with boundary $\partial\Omega$. Moreover, $\lambda \in \mathbb{R}$ and

$$\mathcal{A} := -\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j + \sum_{j=1}^{n} a_j\partial_j + a_0$$

is uniformly strongly elliptic with

$$a_{ij} = a_{ji} \in C(\overline{\Omega})\ , \qquad a_j, a_0 \in L_\infty(\Omega)\ , \quad 1 \le i,j \le n\ .$$

(Throughout the main body of this paper all functions are real-valued.)

1991 *Mathematics Subject Classification.* 35J65, 35J45, 47H07, 47H10, 47H15.

Key words and phrases. Superlinear elliptic boundary value problems, positive solutions, maximum principles, a priori estimates.

We denote by Γ_0 and Γ_1 two disjoint open and closed subsets of $\partial\Omega$ with $\Gamma_0 \cup \Gamma_1 = \partial\Omega$ and put

$$\mathcal{B}u := \begin{cases} u & \text{on } \Gamma_0 \ , \\ \partial_\beta u + b_0 u & \text{on } \Gamma_1 \ , \end{cases}$$

where $\beta \in C^1(\Gamma_1, \mathbb{R}^n)$ is an outward pointing, nowhere tangent vector field and $b_0 \in C(\Gamma_1)$ is nonnegative. Thus $\mathcal{B}$ is the Dirichlet boundary operator on Γ_0, and the Neumann or a first order regular oblique derivative boundary operator on Γ_1. Of course, either Γ_0 or Γ_1 may be empty.

We suppose that a is a bounded measurable function on $\overline{\Omega}$ and put

$$\Omega_\pm := \{ x \in \Omega \ ; \ a^\pm(x) > 0 \} \ ,$$

where $a^+ := \max\{a, 0\}$ is the positive, and $a^- := a^+ - a$ is the negative part of a. Then we assume that

$$\left.\begin{aligned} &\Omega_+ \text{ and } \Omega_- \text{ are open and of class } C^2 \text{ and} \\ &a^\pm \text{ is bounded away from zero on compact subsets of } \Omega_\pm \ . \end{aligned}\right\} \tag{1.2}$$

Note that Ω_+ and Ω_- have only finitely many components. If Γ is any of the components of Γ_1 then we require that

$$\Gamma \cap \partial\Omega_i \neq \emptyset \quad \Rightarrow \quad \Gamma \subset \partial\Omega_i \ , \qquad i \in \{+, -\} \ . \tag{1.3}$$

As for the nonlinearity, we suppose that

$$f \in C\big((\overline{\Omega}_+ \cup \overline{\Omega}_-) \times \mathbb{R}^+, \mathbb{R}^+\big) \ , \qquad f(\cdot, 0) = 0 \ ,$$

where $\mathbb{R}^+ := [0, \infty)$. Then, denoting by $f_\pm$ the restriction of f onto $\overline{\Omega}_\pm \times \mathbb{R}^+$, we assume that the derivative of $f_\pm$ with respect to ξ, denoted by $\partial f_\pm$, exists on $\overline{\Omega}_\pm \times (0, \infty)$ and is continuous, and

$$\left.\begin{aligned} &\partial f_-(x, \xi) > 0 \text{ for} (x, \xi) \in \Omega_- \times (0, \infty) \ , \\ &\lim_{\xi \to \infty} f_-(x, \xi) = \infty, \text{ uniformly for } x \text{ in compact subsets of } \Omega_- \ . \end{aligned}\right\} \tag{1.4}$$

Moreover,

$$\left.\begin{aligned} &f_+(\cdot, \xi) > 0 \text{ for } \xi \in (0, \infty); \quad r \in (1, \infty); \text{ and } \ell \text{ is a bounded positive} \\ &\text{function on } \overline{\Omega}_+, \text{ which is bounded away from zero, such that} \\ &\lim_{\xi \to \infty} \xi^{1-r} f_+(x, \xi) = \ell(x), \text{ uniformly for } x \in \Omega_+ \ . \end{aligned}\right\} \tag{1.5}$$

(Here and in the following we use the point-wise order for real-valued functions, and we write $g > h$ if $g \geq h$ and $g \neq h$.)

Note that $f(x, \xi) = \xi^{r-1}$ satisfies the above hypotheses, and in this case the nonlinearity in (1.1) equals $a(x)u^r$.

Our setting is wide enough to include purely sublinear problems ($\Omega_+ = \emptyset$) and purely superlinear equations ($\Omega_- = \emptyset$). The most general situation occurs, of course, if both Ω_+ and Ω_- are nonempty. In this case one speaks of superlinear indefinite problems.

Semilinear elliptic boundary value problems of type (1.1) have attracted a great deal of interest during the last few decades. Most of the published research deals, however, either with the purely sublinear or with the purely superlinear case, where — in addition — a is not allowed to vanish on sets of positive measure, in general. A substantial amount of the literature in this field concerns self-adjoint problems that can be analyzed by variational techniques.

The simplest case is $\Omega_- = \Omega$ and it has been studied by many authors. Large positive constants provide supersolutions, and there is at most one positive solution.

A more difficult case occurs if $\Omega_+ = \emptyset$ and $\Omega_- \neq \Omega$, since now the a priori bounds do not prevail. In the particular situation where $\mathcal{A} := -\Delta$ and $\Gamma_1 = \emptyset$ existence and uniqueness of positive solutions have been established in [BO] by variational methods. The corresponding Neumann problem, that is, $\mathcal{A} := -\Delta$ and $\Gamma_0 = \emptyset$ with $\mathcal{B} = \partial_\nu$, where ν is the outer unit normal on $\partial\Omega$, but for a less general class of nonlinearities, has been studied in [O2] by continuation methods. The results of [BO] have been extended in [FKLM] to not necessarily self-adjoint problems with smooth coefficients under rather general boundary conditions by means of the method of sub- and supersolutions.

Classical papers devoted to the superlinear case $\Omega_+ = \Omega$ are [AR], [BT], [FLN], [GS2], [N], [Po], and [Tu]. In [BT] it was shown that there exist positive solutions under Dirichlet boundary conditions (that is, $\Gamma_1 = \emptyset$), provided $r < (n+1)/(n-1)$ and $\lambda \geq 0$, and $\mathcal{A}$ satisfies the strong maximum principle. The proof relies on the existence of a priori bounds and positive operator theory. In [BT] substantial generalizations of the earlier theorems in [N] and [Tu] are achieved. The results of [Po] and [AR], which are obtained by variational techniques, as well as an analysis of the radially symmetric case suggest that a priori bounds for positive solutions should exist for $r < (n+2)/(n-2)$ if $n \geq 3$ (cf. [A2, Section 22], [BT]). Indeed, it was in [GS2] where the existence of a priori bounds for all positive solutions was established for all $r > 1$ if $n = 2$, and for $r < (n+2)/(n-2)$ if $n \geq 3$. The proof consists of an indirect argument using a scaling technique which reduces the equation to a Liouville type problem. About simultaneously, the same result had also been obtained in [FLN] by exploiting the symmetry properties of the Laplace operator.

Relatively little is known in the case of indefinite superlinear problems. The Neumann problem for $\mathcal{A} = -\Delta$ has been studied in [O2] for $f(\cdot,\xi) = \xi^{r-1}$. By means of variational techniques it was shown that there exist two positive solutions for each λ belonging to some interval $(0, \lambda_0)$, provided $r < (n+2)/(n-2)$ and $n \geq 3$. This paper also contains a priori bounds for positive solutions. However, these bounds do not seem to be valid for all positive solutions.

In [AT] the authors investigate the Dirichlet problem for $\mathcal{A} = -\Delta$ by means of variational techniques for $f_+(\cdot,\xi) = \xi^{r-1}$ and $f_-(\cdot,\xi) = \xi^{p-1}$ where $1 < r < p$ and $r < (n+2)/(n-2)$ if $n \geq 3$. They establish the existence of numbers σ and λ_0 with $\sigma < \lambda_0$ such that (1.1) has for each $\lambda \in (\sigma, \lambda_0)$ at least two positive solutions and no positive solution for $\lambda > \lambda_0$.

More general classes of equations have been handled in [BCN2] and in [L2]. These results are local inasmuch as they can be obtained by local bifurcation and implicit function arguments, although in [BCN2] variational techniques have been employed.

The paper which has strongly motivated our research is [BCN1]. Besides of imposing more regularity on $\mathcal{A}$ and restricting the class of admissible boundary operators the authors of [BCN1] assume that

$$\Omega_+ \neq \emptyset\ , \quad \Omega_- = \Omega\backslash\overline{\Omega}_+ \neq \emptyset\ , \quad \overline{\Omega}_+ \cap \overline{\Omega}_- \subset \Omega\ ,$$

and

$$\nabla a(x) \neq 0\ , \qquad x \in \overline{\Omega}_+ \cap \overline{\Omega}_-\ .$$

Thus a full neighborhood for $\partial\Omega$ belongs either to Ω_+ or to Ω_-, and a is not allowed to vanish on a set of positive measure. Under these assumptions the main result of [BCN1] guarantees that (1.1) has a positive solution provided $\lambda = 0$ and

$$1 < r < (n+2)/(n-1) \tag{1.6}$$

and the principal eigenvalue, σ^Ω, of $(\mathcal{A},\mathcal{B})$ is positive. The proof of this theorem is based on a priori bounds for positive solutions and Leray-Schauder degree arguments. The a priori bounds are established by adequate adaptions of the rescaling argument of [GS1] and a new Liouville type theorem for semilinear elliptic equations in cones.

It is one of the goals of our paper to give an extension of the main result of [BCN1]. More precisely, we show that if there exists a constant $\gamma \geq 0$ such that

$$a^+(x) \sim \big[\mathrm{dist}(x,\partial\Omega_+)\big]^\gamma \qquad \text{near } \partial\Omega_+\ , \tag{1.7}$$

and if

$$r < (n+1+\gamma)/(n-1) \tag{1.8}$$

and

$$r < (n+2)/(n-2) \qquad \text{for } n \geq 3\ , \tag{1.9}$$

then the positive solutions of (1.1) are bounded in $C(\overline{\Omega})$ if λ stays bounded (cf. Theorem 4.3). Moreover, if we denote by Λ the set of λ-values for which (1.1) has a positive solution, then either $\Lambda = (-\infty,\sigma^\Omega)$ or $\Lambda = (-\infty,\lambda^*]$ for some $\lambda^* > \sigma^\Omega$ (Theorem 7.1). In the latter case (1.1) has at least two solutions for $\sigma^\Omega < \lambda < \lambda^*$ (Theorem 7.4). Observe that, even in the case where $\gamma = 1$, arising if $\nabla a^+ \neq 0$ on $\partial\Omega_+$, this result is a substantial generalization of the main theorem of [BCN1], not only since it guarantees the existence of multiple solutions but also due to the absence of further restrictions on Ω_+.

Our proof adapts the rescaling arguments of [GS2] and [BCN1] and relies on the crucial new observation that positive solutions are bounded in $C(\overline{\Omega})$ if they are bounded in $C(\overline{\Omega}_+)$ (Theorem 4.1). The latter fact is derived by investigating the growth of the positive solutions of the underlying sublinear problem if λ approaches the point where bifurcation from infinity occurs (Section 3), and on a characterization of the strong maximum principle by the existence of positive strict supersolutions (Theorem 2.4).

Observe that (1.9) implies (1.8) if $n \geq 3$ and $\gamma \geq 2n/(n-2)$. Thus in this case we get a priori bounds for positive solutions in the range $1 < r < (n+2)/(n-2)$, which is optimal. In particular, we extend in this case the multiplicity results of [O2] and [AT] to our general setting which is not variational.

The smallest range for r for which we establish a priori bounds occurs when $\gamma = 0$, that is, if a^+ is bounded away from zero on Ω_+. In this limiting case r has to satisfy the restriction $1 < r < (n+1)/(n-1)$ which is the bound of [BT] (where no assumptions on the decay of a^+ near $\partial\Omega_+$ have been made).

A second goal of this paper is the derivation of a priori bounds without imposing a restriction of the form (1.7). This is achieved by employing the weak Harnack inequality on Ω^+, interior L_p-estimates, and bootstrapping arguments, provided $\overline{\Omega}_+ \subset \Omega$ and $\overline{\Omega}_+ \cap \overline{\Omega}_- = \emptyset$. This leads to existence and multiplicity results if

$$r < n/(n-2) \tag{1.10}$$

for $n \geq 3$ (Theorems 5.2 and 7.4). Since $(n+1)/(n-1) < n/(n-2)$ our result improves the main theorem of [BT]. Moreover, (1.10) is less restrictive than (1.8) if a^+ satisfies (1.7) and $\gamma < 2/(n-2)$.

This paper has three parts and an appendix. In the first part, which consists of Section 2, we give a characterization of the strong maximum principle for our general elliptic boundary value problem $(\mathcal{A}, \mathcal{B})$ by means of the existence of positive strict supersolutions and the positivity of the principal eigenvalue. This characterization extends the one of [L1], where Dirichlet boundary conditions have been considered, to the case of the general boundary operator $\mathcal{B}$, where we emphasize that there is no sign restriction on b_0. The results of this section are the basis for showing that the usual monotonicity and comparison theorems, which are known to hold for Dirichlet boundary conditions and which are used throughout this paper, can be extended to our general setting.

In the second part we establish a priori bounds for sets of positive solutions of (1.1) under various restrictions, some of which we have described above.

In the third part which comprises Section 7 we deduce existence and multiplicity results for positive solutions of (1.1) by employing monotonicity and bifurcation techniques as well as fixed point methods in ordered Banach spaces (cf. [A2]).

In the appendix we include a proof of interior L_p-estimates for elliptic equations under minimal smoothness assumptions on the coefficients (Theorem A2.1). In addition, we show how these estimates can be used to improve a priori bounds for families of semilinear elliptic equations (Theorem A3.1). These results are used in the proofs of Theorems 5.2 and 6.1. Since interior L_p-estimates under minimal smoothness hypotheses are of independent interest and since we could not find them in the literature in the form which is needed in this paper we have decided to derive them for rather general elliptic systems of arbitrary order.

2 A Characterization of the Strong Maximum Principle. In [L1, Theorem 2.5] the strong maximum principle for second order elliptic equations has been characterized in the case of Dirichlet boundary conditions by the existence of positive strict supersolutions. In this section we extend that characterization to boundary conditions of the form $\mathcal{B}u = 0$. For this we rely on an inverse positivity result and an existence theorem for the principal eigenvalue given in [A3]. In this section we do not impose a sign restriction on b_0.

In the following we use the natural product order on $L_p(\Omega) \times L_p(\partial\Omega)$. Recall that $p > n$ implies $W_p^2(\Omega) \hookrightarrow C^{2-n/p}(\overline{\Omega})$ and that each $u \in W_p^2(\Omega)$ is a.e. in Ω twice classically differentiable (e.g., [St, Theorem VIII.1]).

Supose that $p > n$. Then $u \in W_p^2(\Omega)$ is said to be **strongly positive** if $u(x) > 0$ for $x \in \Omega \cup \Gamma_1$ and $\partial_\alpha u(x) < 0$ for $x \in \Gamma_0$ with $u(x) = 0$ and any outward pointing, nowhere tangent vector field α on Γ_0. Finally, $(\mathcal{A}, \mathcal{B}, \Omega)$ is said to satisfy the

strong maximum principle if $p > n$, $u \in W_p^2(\Omega)$, and $(\mathcal{A}u, \mathcal{B}u) > 0$ imply that u is strongly positive.

Using this definition we can formulate Theorem 6.1 of [A1] as follows:

2.1 Theorem. *There exists $\omega_0 \in \mathbb{R}$ such that $(\mathcal{A}+\omega, \mathcal{B}, \Omega)$ satisfies for each $\omega > \omega_0$ the strong maximum principle.*

Suppose that $p > n$ and consider the eigenvalue problem

$$\mathcal{A}u = \sigma u \quad \text{in } \Omega \,, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

in $W_p^2(\Omega)$. Putting $W_{p,\mathcal{B}}^2(\Omega) := \left\{ u \in W_p^2(\Omega) \ ; \ \mathcal{B}u = 0 \right\}$ and $A_p := \mathcal{A} \,|\, W_{p,\mathcal{B}}^2(\Omega)$, considered as an unbounded linear operator in $L_p(\Omega)$ with dense domain $W_{p,\mathcal{B}}^2(\Omega)$, problem (2.1) can be reformulated as the eigenvalue equation $A_p u = \sigma u$ in $L_p(\Omega)$. It is an easy consequence of standard regularity theory that the spectrum and the eigenspaces of A_p are independent of $p > n$.

2.2 Theorem. *There exists a least real eigenvalue of* (2.1)*, denoted by $\sigma^\Omega(\mathcal{A}, \mathcal{B})$ and called* **principal eigenvalue** *of $(\mathcal{A}, \mathcal{B}, \Omega)$. It is simple and possesses a unique normalized positive eigenfunction, the* **principal eigenfunction** *of $(\mathcal{A}, \mathcal{B}, \Omega)$. It is strongly positive and $\sigma^\Omega(\mathcal{A}, \mathcal{B})$ is the only eigenvalue of* (2.1) *possessing a positive eigenfunction. Any other eigenvalue σ of* (2.1) *satisfies $\operatorname{Re}\sigma > \sigma^\Omega(\mathcal{A}, \mathcal{B})$, and $(\omega + A_p)^{-1} \in \mathcal{L}\big(L_p(\Omega)\big)$ is positive, compact, and irreducible for $\omega > \sigma^\Omega(\mathcal{A}, \mathcal{B})$.*

Proof. This is Theorem 12.1 of [A1]. In the proof of that theorem it has been referred to [S, appendix 3.2] to assert that a positive compact irreducible linear operator on a Banach lattice has a strictly positive spectral radius. However, this does not follow from the results in [S] but is Theorem 3 in [P]. $\square$

If $p > n$ then $\overline{u} \in W_p^2(\Omega)$ is said to be a **positive strict supersolution** for $(\mathcal{A}, \mathcal{B}, \Omega)$, provided $\overline{u} \geq 0$ and $(\mathcal{A}\overline{u}, \mathcal{B}\overline{u}) > 0$.

2.3 Lemma. *Suppose that $p > n$ and $\overline{u} \in W_p^2(\Omega)$ is a positive strict supersolution for $(\mathcal{A}, \mathcal{B}, \Omega)$. Then $\overline{u}$ is strongly positive.*

Proof. Fix $\omega > 0 \vee \omega_0$. Then $\big((\mathcal{A} + \omega)\overline{u}, \mathcal{B}\overline{u}\big) > 0$, and Theorem 2.1 implies the assertion. $\square$

After these preparations we can easily prove the announced characterization of the maximum principle.

2.4 Theorem. *The following assertions are equivalent*:

(i) $\sigma^\Omega(\mathcal{A}, \mathcal{B}) > 0$*;*
(ii) $(\mathcal{A}, \mathcal{B}, \Omega)$ *possesses a positive strict supersolution;*
(iii) $(\mathcal{A}, \mathcal{B}, \Omega)$ *satisfies the strong maximum principle.*

Proof. (i)⇒(ii): In this case φ is a positive strict supersolution for $(\mathcal{A}, \mathcal{B}, \Omega)$.

(ii)⇒(iii): Suppose that $p > n$ and $u \in W_p^2(\Omega)$ satisfies $(\mathcal{A}u, \mathcal{B}u) > 0$ and $u \not\geq 0$. By assumption there exist $q > n$ and a positive strict supersolution $\overline{u} \in W_q^2(\Omega)$ for $(\mathcal{A}, \mathcal{B}, \Omega)$. Since $W_{p_1}^2(\Omega) \hookrightarrow W_{p_2}^2(\Omega)$ for $p_1 > p_2$, we can assume, by replacing p or q by $p \wedge q$, that $\overline{u} \in W_p^2(\Omega)$. Since $\overline{u}$ is strongly positive, there exists $t > 0$ such that $t\overline{u} + u \geq 0$. Denote by $\overline{t}$ the minimum of all these numbers and note that $\overline{t} > 0$. Then $\overline{t}\overline{u} + u$ is a positive strict supersolution for $(\mathcal{A}, \mathcal{B}, \Omega)$, hence strongly positive. From this we easily infer that there exists $s \in (0, \bar{t})$ with $s\bar{u} + u \geq 0$, which contradicts the definition of $\bar{t}$. Thus $u \geq 0$ and $(\mathcal{A}, \mathcal{B}, \Omega)$ satisfies the strong maximum principle.

(iii)$\Rightarrow$(i): This is an easy consequence of the Krein-Rutman theorem. $\square$

From Theorem 2.4 and by means of the arguments of [L1] we can obtain all the comparison and monotonicity properties of the principal eigenvalues that we use in this paper. Moreover, the proof of Theorem 4.2 of [L1] can easily be modified to yield that $\sigma^\Omega(\mathcal{A}, \mathcal{B})$ depends continuously on Ω if we perturb Ω in such a way that Γ_1 is kept fixed.

3 Necessary Conditions For the Existence of Positive Solutions. Let Ω' be an open set of class C^2 contained in Ω such that, given any component Γ of $\partial\Omega$,

$$\Gamma \cap \partial\Omega' \neq \emptyset \quad \Rightarrow \quad \Gamma \subset \partial\Omega' \ .$$

Then Ω' is said to be **regular**. In this case we put

$$\mathcal{B}_{\Omega'} u := \begin{cases} u & \text{on } \partial\Omega' \cap \Omega \ , \\ \mathcal{B}u & \text{on } \partial\Omega' \cap \partial\Omega \ . \end{cases}$$

Then the results of Section 2 guarantee that the principal eigenvalue $\sigma^{\Omega'}(\mathcal{A}, \mathcal{B}_{\Omega'})$ of $(\mathcal{A}, \mathcal{B}_{\Omega'}, \Omega')$ is well-defined.

We assume throughout that

$$D := \Omega \backslash \overline{\Omega}_- \quad \text{is regular} \ . \tag{3.1}$$

Then D possesses a finite number of components $D_1, \ldots, D_N$, if it is not empty. In the latter case we define the **principal eigenvalue** of D by

$$\sigma^D := \min_{1 \leq j \leq N} \sigma^{D_j}(\mathcal{A}, \mathcal{B}_{D_j}) \ ,$$

and we assume without loss of generality that $\sigma^D = \sigma^{D_1}(\mathcal{A}, \mathcal{B}_{D_1})$. Using these notations and conventions we begin by considering an auxiliary problem. Here and in the following we put

$$\sigma^\Omega := \sigma^\Omega(\mathcal{A}, \mathcal{B})$$

for abbreviation.

3.1 Theorem. *If $\Omega_- \neq \emptyset$ then*

$$\begin{aligned} \mathcal{A}u &= \lambda u - a^- f_-(\cdot, u)u && \text{in } \Omega \ , \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega \ , \end{aligned} \tag{3.2}$$

has a positive solution iff $\sigma^\Omega < \lambda < \sigma^D$. If this is the case, there exists exactly one positive solution, denoted by θ_λ, and the map

$$(\sigma^\Omega, \sigma^D) \to C(\overline{\Omega}) \ , \quad \lambda \mapsto \theta_\lambda \tag{3.3}$$

is C^1, increasing, and $\theta_\lambda \to 0$ in $C(\overline{\Omega})$ as $\lambda \to \sigma^\Omega$.

Proof. This follows easily by adapting the arguments of [FKLM] and [L1]. $\square$

The next proposition describes the behavior of (3.3) at the right endpoint of its interval of existence.

3.2 Proposition. $\lim_{\lambda\to\sigma^D}\theta_\lambda=\infty$, *uniformly on compact subsets of* D_1.

Proof. By differentiating (3.3) with respect to λ we find, for any fixed $\lambda\in(\sigma^\Omega,\sigma^D)$,

$$\big(\mathcal{A}+a^-\theta_\lambda\partial f_-(\cdot,\theta_\lambda)+a^-f_-(\cdot,\theta_\lambda)-\lambda\big)\dot\theta_\lambda=\theta_\lambda\quad\text{in }\Omega\ ,$$

and $\mathcal{B}\dot\theta_\lambda=0$ on $\partial\Omega$, where $\dot\theta_\lambda$ denotes the derivative of θ_λ with respect to λ. Since a^- vanishes on D_1 it follows that

$$(\mathcal{A}-\lambda)\dot\theta_\lambda=\theta_\lambda\quad\text{in }D_1\ ,\qquad \mathcal{B}_{D_1}\dot\theta_\lambda\geq 0\quad\text{on }\partial D_1\ .$$

Let φ_1 be the principal eigenfunction of $(\mathcal{A},\mathcal{B}_{D_1},D_1)$. Then there exists $c_0>0$ such that $\theta_\lambda>c_0\varphi_1$ on D_1. Since $\lambda<\sigma^{D_1}$, Theorem 2.4 implies that $(\mathcal{A}-\lambda,\mathcal{B}_{D_1},D_1)$ satisfies the strong maximum principle. Hence

$$\dot\theta_\lambda\geq(A-\lambda)^{-1}c_0\varphi=\frac{c_0}{\sigma^D-\lambda}\varphi_1\quad\text{on }D_1\ ,$$

where we write A for A_p if it is irrelevant which $p>n$ is being considered. Now the assertion follows from the fact that φ_1 is bounded away from zero on every compact subset of D_1. $\square$

We turn to the study of problem (1.1) for a fixed $\lambda\in\mathbb{R}$, that is, to

$$\begin{aligned}\mathcal{A}u&=\lambda u+af(\cdot,u)u &&\text{in }\Omega\ ,\\ \mathcal{B}u&=0 &&\text{on }\partial\Omega\ .\end{aligned}\tag{3.4$_\lambda$}$$

Henceforth we presuppose that

$$\Omega_+\neq 0$$

since the case $\Omega_+=0$ is covered by Theorem 3.1. First we prove a nonexistence result.

3.3 Theorem. *Problem* $(3.4)_\lambda$ *does not have positive solutions if* $\lambda\geq\sigma^D$.

Proof. Let u be a positive solution of $(3.4)_\lambda$ for some $\lambda\in\mathbb{R}$. Then

$$\big(\mathcal{A}-af(\cdot,u)\big)u=\lambda u\quad\text{in }\Omega\ ,\qquad \mathcal{B}u=0\quad\text{on }\partial\Omega\ ,$$

and the uniqueness result for the principal eigenvalue contained in Theorem 2.2 guarantees that

$$\lambda=\sigma^\Omega\big(\mathcal{A}-af(\cdot,u),\mathcal{B}\big)\ .\tag{3.5}$$

Since a^- vanishes on D it follows from the monotonicity of the principal eigenvalue that

$$\lambda<\sigma^D\big(\mathcal{A}-a^+f(\cdot,u),\mathcal{B}_D\big)\leq\sigma^D(\mathcal{A},\mathcal{B}_D)=\sigma^D\ ,$$

thanks to $f(x,\xi)>0$ for $x\in\Omega_+$ and $\xi>0$. $\square$

Next we establish a technical estimate which will also be useful in Section 7.

3.4 Lemma. *Let u be a positive solution of* $(3.4)_\lambda$ *for some* $\lambda < \sigma^D$ *and let* Q *be open and of class* C^2 *with* $\overline{Q} \subset \Omega_+$*. Then*

$$\min_{\overline{Q}} f_+\big(\cdot, u(\cdot)\big) < \frac{\sigma^Q(\mathcal{A}, \mathcal{B}_Q) - \lambda}{\inf_Q a^+} .$$

Proof. Since $\overline{Q} \subset \Omega_+ \subset D$ we infer from the monotonicity properties of the principal eigenvalue that $\lambda < \sigma^D < \sigma^Q(\mathcal{A}, \mathcal{B}_Q)$. Moreover, from $a^- \,|\, \Omega_+ = 0$ and (3.5) it follows that

$$\lambda < \sigma^Q\big(A - a^+ f_+(\cdot, u), \mathcal{B}_Q\big) < \sigma^Q(\mathcal{A}, \mathcal{B}_Q) - \inf_Q a^+ \min_{\overline{Q}} f_+(\cdot, u) ,$$

which proves the assertion. □

Using these results we establish a sufficient condition for $(3.4)_\lambda$ not to have positive solutions for λ in a neighborhood of σ^D.

3.5 Theorem. *Suppose that* $D_1 \cap \Omega_+ \neq \emptyset$*. Then there exists* $\lambda^* < \sigma^D$ *such that* $(3.4)_\lambda$ *does not have a positive solution for* $\lambda > \lambda^*$*.*

Proof. Let u_λ be a positive solution of $(3.4)_\lambda$ for some $\lambda \in (\sigma^\Omega, \sigma^D)$. Then u_λ is a supersolution of (3.2). Fix $\omega \geq 0$ such that $\omega + \sigma^\Omega > 0$ and add ωu on both sides of the first equation in (3.2). Then $(\mathcal{A} + \omega, \mathcal{B}, \Omega)$ satisfies the strong maximum principle by Theorem 2.4, and we infer that $u_\lambda \geq \theta_\lambda$. Let Q be an open ball with $\overline{Q} \subset D_1 \cap \Omega_+$ and fix $\varepsilon > 0$ such that $\ell(x) \geq 2\varepsilon$ for a.a. $x \in \Omega_-$. Then Proposition 3.2 and (1.5) guarantee the existence of $\lambda^* < \sigma^D$ such that

$$u_\lambda(x) \geq \theta_\lambda(x) \geq \Big(\frac{\sigma^Q(\mathcal{A}, \mathcal{B}_Q) - \sigma^\Omega}{\varepsilon \inf_Q a^+}\Big)^{1/(r-1)} \tag{3.6}$$

and

$$f\big(x, u_\lambda(x)\big) \geq \big(\ell(x) - \varepsilon\big) u(x)^{r-1} \geq \varepsilon u(x)^{r-1}$$

for $x \in \overline{Q}$ and any $\lambda \in [\lambda^*, \sigma^D)$ for which $(3.4)_\lambda$ has a positive solution u_λ. Hence we deduce from Lemma 3.4 that

$$\min_{\overline{Q}} u^{r-1} \leq \frac{\sigma^Q(\mathcal{A}, \mathcal{B}_Q) - \sigma^\Omega}{\varepsilon \inf_Q a^+} ,$$

which contradicts (3.6). Thus $(3.4)_\lambda$ cannot have a positive solution for $\lambda > \lambda^*$. □

3.6 Remark. If D is nonempty and connected and Ω_+ is nonempty as well then the hypothesis of Theorem 3.5 is satisfied. □

4 A Priori Bounds by Scaling Arguments. In the following (u, λ) is said to be a positive solution of (1.1) if u is a positive solution of $(3.4)_\lambda$.

We begin by establishing a prior bounds for sets of positive solutions of (1.1) which will be useful for deriving existence and multiplicity results in later sections. Our first theorem shows that uniform a priori bounds on Ω_+ imply uniform bounds on Ω.

4.1 Theorem. *Let $\mathcal{S}$ be a set of positive solutions of* (1.1) *such that*

$$\Lambda_{\mathcal{S}} := \left\{ \lambda \in \mathbb{R} \; ; \; (u,\lambda) \in \mathcal{S} \right\}$$

is bounded. Then

$$\sup_{(u,\lambda)\in\mathcal{S}} \sup_{\Omega_+} u < \infty \quad \Rightarrow \quad \sup_{(u,\lambda)\in\mathcal{S}} \sup_{\Omega} u < \infty \; , \tag{4.1}$$

provided $\Omega_+ \neq \emptyset$.

Proof. Fix $(u,\lambda) \in \mathcal{S}$. Then $\lambda < \sigma^D$ by Theorem 3.3. Since

$$\Omega_0 := \Omega \backslash (\overline{\Omega}_+ \cup \overline{\Omega}_-) \tag{4.2}$$

is a regular open subset of Ω, properly contained in D, we see that $\lambda < \sigma^{\Omega_0}(\mathcal{A}, \mathcal{B}_{\Omega_0})$.

Put

$$\Omega_\delta := \Omega_0 \cup \left\{ x \in \Omega_- \; ; \; d(x,\Omega_-) < \delta \right\} \cup (\partial\Omega_- \cap \Omega) \backslash \partial\Omega_+$$

for $\delta > 0$. Then Ω_δ is for each sufficiently small δ a regular open subset of Ω, and $\Omega_\delta \downarrow \Omega_0$ in the sense of [L1]. Since the boundaries of distinct Ω_δ differ only where $\mathcal{B}_{\Omega_\delta}$ reduce to Dirichlet boundary operators it follows that

$$\lim_{\delta\to 0} \sigma^{\Omega_\delta}(\mathcal{A}, \mathcal{B}_{\Omega_\delta}) = \sigma^{\Omega_0}(\mathcal{A}, \mathcal{B}_{\Omega_0}) \; .$$

In fact, the family $\{ \Omega_\delta \; ; \; \delta > 0 \}$ can by obtained from Ω by a parameter-dependent holomorphic family of diffeomorphisms and hence the previous relation follows from the theory described in Chapter VII of [K]. Thus we can fix $\delta > 0$ such that $\lambda < \sigma^{\Omega_\delta}(\mathcal{A}, \mathcal{B}_{\Omega_\delta})$, so that $(\mathcal{A} - \lambda, \mathcal{B}_{\Omega_\delta}, \Omega_\delta)$ satisfies the strong maximum principle, thanks to Theorem 2.4.

Denote by M the supremum on the left-hand side of (4.1) and let ψ be the unique solution of

$$(\mathcal{A} - \lambda)v = 0 \quad \text{in } \Omega_\delta \; , \qquad \mathcal{B}_{\Omega_\delta} v = \begin{cases} M & \text{on } \partial\Omega_\delta \cap (\partial\Omega_+ \backslash \partial\Omega) \; , \\ 0 & \text{on } \partial\Omega_\delta \backslash (\partial\Omega_+ \backslash \partial\Omega) \; . \end{cases}$$

Fix $p > n$ and denote by $w \in W_p^2(\Omega)$ an extension of $\psi \,|\, \Omega_{\delta/2}$ with $\min_{\overline{\Omega}} w > 0$. Then we claim that kw is for sufficiently large $k > 0$ a positive strict supersolution of

$$\begin{aligned} \mathcal{A}v &= \lambda v - a^- f(\cdot, v)v && \text{in } \Omega \backslash \overline{\Omega}_+ \; , \\ v &= M && \text{on } \partial\Omega_+ \backslash \partial\Omega \; , \\ \mathcal{B}v &= 0 && \text{on } \partial(\Omega \backslash \overline{\Omega}_+) \cap \partial\Omega \; , \end{aligned} \tag*{$(4.3)_\lambda$}$$

for each $\lambda \in \Lambda_{\mathcal{S}}$. Indeed, in $\Omega_{\delta/2}$ we have

$$\mathcal{A}(kw) = k\mathcal{A}w = k\lambda w > k\lambda w - a^- kw f_-(\cdot, kw)$$

for each $k > 0$, thanks to the fact that $f(\cdot, 0) = 0$ and (1.4) imply $f_-(\cdot, \xi) > 0$ for $\xi > 0$. On $\Sigma_\delta := \left\{ x \in \Omega_- \; ; \; d(x, \partial\Omega_-) \geq \delta/2 \right\}$ the functions a^- and w are positive and bounded away from zero, and $f(x,\xi) \to \infty$ as $\xi \to \infty$, uniformly with respect to x, by (1.4), Thus there exists $k > 0$ such that

$$\mathcal{A}w > \lambda w - a^- f(\cdot, kw)w \ , \qquad \lambda \in \Lambda_{\mathcal{S}} \ ,$$

on Σ_δ. Moreover, on $\partial(\Omega\backslash\Omega_+) \cap \partial\Omega$ the operator $\mathcal{B}_{\Omega_\delta}$ coincides with $\mathcal{B}$ and w equals ψ so that $\mathcal{B}w = 0$ there. Finally, on $\partial\Omega_+\backslash\partial\Omega$ we know that w is bounded away from zero. Thus $\overline{u} := kw$ is, indeed, a positive strict supersolution for $(4.3)_\lambda$, independently of $\lambda \in \Lambda_{\mathcal{S}}$, provided $k > 0$ is sufficiently large.

If $(u,\lambda) \in \mathcal{S}$ then it follows that $v := \overline{u} - u$ satisfies

$$\begin{aligned} \big(\mathcal{A} - \lambda + a^- g(\overline{u},u)\big)w &\geq 0 \qquad && \text{in } \Omega\backslash\overline{\Omega}_+ \ , \\ w &> 0 && \text{on } \partial\Omega_+\backslash\partial\Omega \ , \\ \mathcal{B}w &= 0 && \text{on } \partial(\Omega\backslash\overline{\Omega}_+) \cap \partial\Omega \ , \end{aligned} \tag{4.4$_\lambda$}$$

where

$$g(\overline{u},u) := f_-(\cdot,\overline{u}) + \int_0^1 \partial f_-(\cdot, u + tv)u\,dt \geq f_-(\cdot,\overline{u}) \ ,$$

thanks to (1.4). Hence

$$\sigma^{\Omega\backslash\overline{\Omega}_+}\big(\mathcal{A} - \lambda + a^- g(\overline{u},u), \mathcal{B}_{\Omega\backslash\overline{\Omega}_+}\big) \geq \sigma^{\Omega\backslash\overline{\Omega}_+}\big(\mathcal{A} - \lambda + a^- f(\cdot,\overline{u}), \mathcal{B}_{\Omega\backslash\overline{\Omega}_+}\big) > 0 \ ,$$

where the last inequality sign follows from Theorem 2.4 and the fact that $\overline{u}$ is a positive strict supersolution for $(4.3)_\lambda$. Thus, by invoking Theorem 2.4 once more, we infer from $(4.4)_\lambda$ that $u \leq \overline{u}$, which implies the assertion. $\square$

Now we derive a priori bounds for positive solutions on Ω^+, provided r satisfies suitable restrictions. For this we first prove a technical result, where we use the scaling arguments of [GS1].

4.2 Lemma. *Suppose that*

$$r < (n+2)/(n-2) \quad \textit{if}\ n \geq 3 \ .$$

Let $\big((u_k,\lambda_k)\big)_{k\in\mathbb{N}}$ *be a sequence of positive solutions of* (1.1) *such that* (λ_k) *is bounded and* $\sup_{\Omega_+} u_k \to \infty$. *Choose* $x_k \in \overline{\Omega}_+$ *with* $u_k(x_k) = \max_{\overline{\Omega}_+} u_k$ *for* $k \in \mathbb{N}$. *Then* $x_k \to \partial\Omega_+$.

Proof. We have to show that each neighborhood of $\partial\Omega_+$ in $\overline{\Omega}_+$ contains all but finitely many of the x_k. Let this be false. Then there exist a compact subset K of Ω_+ and a subsequence, again denoted by (x_k), such that $x_k \in K$ for $k \in \mathbb{N}$ and $x_k \to x_\infty$ for some $x_\infty \in K$.

Put

$$M_k := u_k(x_k) \ , \quad \rho_k := M_k^{(1-r)/2} \ , \qquad k \in \mathbb{N} \ , \tag{4.5}$$

and observe that $r > 1$ and $M_k \to \infty$ imply $\rho_k \to 0$. The change of variables

$$y := (x - x_k)/\rho_k \ , \quad v_k(y) := \rho_k^{2/(r-1)} u(x)$$

transforms the first equation of (1.1) into

$$\mathcal{A}_k v_k = \rho_k^2 \lambda_k v_k + a_k^+ f_k(y, v_k) \ , \tag{4.6}$$

where

$$\mathcal{A}_k := -\sum_{i,j=1}^{n} a_{ij;k}\partial_i\partial_j + \sum_{j=1}^{n} \rho_k a_{j;k}\partial_j + \rho_k^2 a_{0;k} \tag{4.7}$$

with $a_{ij;k}(y) := a_{ij}(x_k + \rho_k y)$ etc., $a_k^+(y) := a^+(x_k + \rho_k y)$, and

$$f_k(y, v_k) := \rho_k^{2r/(r-1)} f_+\bigl(x_k + \rho_k y, \rho_k^{-2/(r-1)} v_k\bigr)\rho_k^{-2/(r-1)} v_k \ ,$$

provided $x_k + \rho_k y \in \Omega_+$. Given any $R > 0$, it follows from $\rho_k \to 0$ and $K \subset\subset \Omega_+$ that there exists k_R such that $x_k + \rho_k y \in \Omega_+$ for each $k \geq k_R$ and all $y \in \mathbb{R}^n$ with $|y| \leq R + 1$. We also see from (4.5) that

$$0 < v_k(y) \leq v_k(0) = \rho_k^{2/(r-1)} M_k = 1 \ , \qquad |y| \leq R + 1 \ , \quad k \geq k_R \ . \tag{4.8}$$

Hence (1.5) implies

$$\lim_{k\to\infty} \frac{f_k(y, v_k(y))}{v_k(y)^r} = \ell(x_\infty) \ , \qquad |y| \leq R + 1 \ . \tag{4.9}$$

Now we infer from (4.6), (4.8), (4.9), and Theorem A2.1 of the appendix that, given any $p > n$, the sequence (v_k) is bounded in $W_p^2(\mathbb{B}_R)$, where $\mathbb{B}_R$ denotes the open ball in $\mathbb{R}^n$ with center at the origin and radius $R > 0$. Thus, by passing to a suitable subsequence, again denoted by (v_k), and by using the compact embedding of $W_p^2(\mathbb{B}_R)$ in $C^1(\overline{\mathbb{B}}_R) \hookrightarrow W_p^1(\mathbb{B}_R) \hookrightarrow L_p(\mathbb{B}_R)$, we can assume that there exists $v \in W_p^2(\mathbb{B}_R)$ such that $v \geq 0$ and v_k converges weakly in $W_p^2(\mathbb{B}_R)$, and strongly in $W_p^1(\mathbb{B}_R)$ and in $C(\overline{\mathbb{B}}_R)$ towards v. From this we easily infer that $(\mathcal{A}_k v_k)$ converges weakly in $L_p(\mathbb{B}_R)$ towards

$$\mathcal{A}_\infty v := -\sum_{i,j=1}^{n} a_{ij}(x_\infty)\partial_i\partial_j v$$

and that $\bigl(\rho_k^2\lambda_k v_k + a_k^+ f_k(\cdot, v_k)\bigr)_{k\in\mathbb{N}}$ converges strongly, hence weakly, in $L_p(\mathbb{B}_R)$ towards $a^+(x_\infty)\ell(x_\infty)v^r$. Consequently,

$$\mathcal{A}_\infty v = a^+(x_\infty)\ell(x_\infty)v^r \quad \text{in } \mathbb{B}_R$$

for each R. By a standard diagonal sequence argument it is not difficult to see that $v \in W_{p,\mathrm{loc}}^2(\mathbb{R}^n)$ and that $\mathcal{A}_\infty v = \alpha v^r$ on $\mathbb{R}^n$, where $\alpha := a^+(x_\infty)\ell(x_\infty) > 0$. Since $v^r \in C^1(\mathbb{R}^n)$, standard elliptic regularity implies that $v \in C^2(\mathbb{R}^n)$. Also note that $v(0) = 1$. Finally, by a linear change of coordinates we find that there exists a nontrivial nonnegative function $w \in C^2(\mathbb{R}^n)$ satisfying $-\Delta w = w^r$, which contradicts Theorem 1.1 of [GS2]. This proves the lemma. □

After these preparations we can derive the desired a priori bounds by arguments similar to the ones used in the proof of Theorem 3.1 of [BCN1].

4.3 Theorem. *Suppose that there exist $\alpha : \overline{\Omega}_+ \to \mathbb{R}^+$, which is continuous and bounded away from zero in a neighborhood of $\partial\Omega_+$, and a constant $\gamma \geq 0$ such that*

$$a^+(x) = \alpha(x)\bigl[\mathrm{dist}(x, \partial\Omega_+)\bigr]^\gamma \ , \qquad x \in \Omega_+ \ . \tag{4.10}$$

Also suppose that

$$r < (n+1+\gamma)/(n-1) \tag{4.11}$$

and

$$r < (n+2)/(n-2) \quad \textit{if } n \geq 3 \ . \tag{4.12}$$

Let $\mathcal{S}$ be a set of positive solutions of (1.1) *such that $\Lambda_{\mathcal{S}}$ is bounded in $\mathbb{R}$. Then $\mathcal{S}$ is bounded in $C(\overline{\Omega}) \times \mathbb{R}$.*

Proof. Let the assertion be false. Then Theorem 4.1 and Lemma 4.2 imply the existence of a sequence $\big((u_k,\lambda_k)\big)_{k\in\mathbb{N}}$ in $\mathcal{S}$, a sequence x_k in Ω_+, and a point $x_\infty \in \partial\Omega_+$ such that $x_k \to x_\infty$ and $M_k := u_k(x_k) = \sup_{\Omega_+} u_k \to \infty$ as $k\to\infty$. Now we define $\rho_k > 0$ by

$$\rho_k^{(2+\gamma)/(r-1)} M_k = 1$$

and transform the first equation in (1.1) by the change of variables

$$y := (x-x_k)/\rho_k \ , \quad v_k(y) := \rho_k^{(2+\gamma)/(r-1)} u_k(x)$$

in

$$\mathcal{A}_k v_k = \rho_k^2 \lambda_k v_k + a_k^+ g_k(\cdot, v_k) \ , \tag{4.13}$$

where $\mathcal{A}_k$ is defined in (4.7) and

$$g_k(y,v_k) := \rho_k^2 f_+\big(x_k + \rho_k y, \rho_k^{-(2+\gamma)/(r-1)} v_k\big) v_k \ , \tag{4.14}$$

provided $x_k + \rho_k y \in \Omega_+$. By an additional change of coordinates, that is independent of $k \in \mathbb{N}$, we can also assume that $\overline{\Omega}_+$ is a neighborhood of 0 in the half-space $\mathbb{H}^n := \{\, x \in \mathbb{R}^n \ ; \ x^n > 0 \,\}$, and that $x_\infty = 0$. Hence, given $R > 0$, there exists k_R such that v_k is well-defined and satisfies (4.13) on

$$H_{R,k} := \mathbb{B}_R \cap \big(-(x_k^n/\rho_k) + \mathbb{H}^n\big)$$

for $k \geq k_R$. Note that $0 < v_k(y) \leq v_k(0) = 1$ and, thanks to (4.10),

$$a_k^+(y) = \rho_k^\gamma \alpha(x_k + \rho_k y)(y^n + x_k^n/\rho_k)^\gamma \tag{4.15}$$

for $k \geq k_R$ and $y \in H_{R,k}$, since $x^n = \operatorname{dist}(x, \partial\Omega_+)$.

(i) Suppose that $(x_k^n/\rho_k)_{k\in\mathbb{N}}$ is not bounded away from zero. By passing to an appropriate subsequence, we can assume that $x_k^n/\rho_k \to 0$ as $k \to \infty$. Then the sequence $(H_{R,k})_{k\in\mathbb{N}}$ approaches $H_R := \mathbb{B}_R \cap \mathbb{H}^n$ and from (1.5) and (4.14) we see that

$$\lim_{k\to\infty} \frac{\rho_k^\gamma g_k(y, v_k(y))}{v_k(y)^r} = \ell_\infty(0) \ , \qquad |y| \leq R \ .$$

Hence we deduce form (4.13) and (4.15), together with the arguments of the proof of Lemma 4.2, based on the corresponding estimates up to the boundary, that there exists a nonnegative $v \in C^2(\overline{\mathbb{H}^n})$ satisfying $v(0) = 1$ and $\mathcal{A}_\infty v = \alpha(0)\ell(0)(y^n)^r v^r$. By an additional suitable linear change of coordinates we see that there exists a nontrivial nonnegative solution of

$$-\Delta u = (x^n)^\gamma u^r \quad \text{in } C^2(\mathbb{H}^n) \ , \tag{4.16}$$

which, thanks to (4.11), contradicts Corollary 2.1 of [BCN1].

(ii) Suppose that $(x_k^n/\rho_k)_{k\in\mathbb{N}}$ is not bounded above. Then, by selecting a suitable subsequence, we can assume that $\beta_k := \rho_k/x_k^n \to 0$ as $k\to\infty$. In this case $(H_{R,k})_{k\in\mathbb{N}}$ approaches $\mathbb{B}_R$ as $k\to\infty$. By introducing the variable $z := y/\beta_k$ equation (4.13) transforms into

$$\widetilde{\mathcal{A}}_k w_k = (\rho_k\beta_k)^2\lambda_k w_k + h_k(z,w_k) \ ,$$

where $\widetilde{\mathcal{A}}_k$ is obtained from $\mathcal{A}_k$ by replacing ρ_k by $\beta_k\rho_k$ everywhere, and

$$h_k(z,w) := \rho_k^{2+\gamma}\alpha(x_k+\rho_k\beta_k z)\big(\beta_k^{1+\frac{2}{\gamma}}z^n+1\big)^\gamma f_+\big(x_k+\rho_k\beta_k z, \rho_k^{-\frac{2+\gamma}{(r-1)}}w\big)w \ ,$$

provided $z\in(1/\beta_k)H_k$. Note that $\beta_k\to 0$ as $k\to\infty$. From this and the arguments of the proof of Lemma 4.2 we infer that there exists $w\in C^2(\mathbb{R}^n)$ with $w\ge 0$ and $w(0)=1$ such that $\mathcal{A}_\infty w = \alpha(0)\ell(0)w^r$. Thus, after a linear coordinate change, we find that there exists a nontrivial nonnegative solution of $-\Delta u = u^r$ in $C^2(\mathbb{R}^n)$. Thanks to (4.12) this contradicts Theorem 1.1 of [GS2].

(iii) Lastly, suppose that $(x_k^n/\rho_k)_{k\in\mathbb{N}}$ is bounded above and bounded away from zero. Then, by choosing a subsequence, if necessary, we can assume that $x_k^n/\rho_k\to s$ for $k\to\infty$ and some $s>0$. Then, by employing the arguments of the proof of Lemma 4.2 once more, we infer from (4.13) and (4.15) the existence of a nonnegative $w\in C^2(-s+\mathbb{H}^n)$ satisfying $\mathcal{A}_\infty w = \alpha(0)\ell(0)(y^n+s)^\gamma w^r$ and $w(0)=1$. Consequently, after an appropriate linear change of coordinates we see that (4.16) has a nontrivial nonnegative solution, which is impossible. This proves the theorem. $\square$

4.4 Remarks. (a) Suppose that $n\ge 3$ and $\gamma\ge 2n/(n-2)$. Then condition (4.12) implies (4.11). Thus, given any set $\mathcal{S}$ of positive solutions of (1.1) such that $\Lambda_{\mathcal{S}}$ is bounded, we see that $\mathcal{S}$ is bounded in $C(\overline{\Omega})\times\mathbb{R}$, provided $1<r<(n+2)/(n-2)$. Observe that $(n+2)/(n-2)$ is the optimal exponent for which we can get uniform a priori bounds for the superlinear problem if $n\ge 3$ (e.g., [FLN], [GS2]).

(b) Suppose that $\gamma<2n/(n-2)$ if $n\ge 3$. Then any set $\mathcal{S}$ of positive solutions of (1.1) such that $\Lambda_{\mathcal{S}}$ is bounded, is bounded in $C(\overline{\Omega})\times\mathbb{R}$, provided

$$r<(n+1+\gamma)/(n-1) \ . \tag{4.17}$$

If $\gamma=1$ then (4.17) reduces to $r<(n+2)/(n-1)$. Under this restriction a corresponding boundedness result for a positive solution of (1.1) has been obtained in [BCN1, Theorem 3.1]. However, the authors of that paper also assume that $a\in C^2(\overline{\Omega})$, that a has a nonvanishing gradient on $\partial\Omega_+\cap\partial\Omega_-$, and that $\Omega_0=\emptyset$.

(c) Suppose that $\mathcal{A}=-\Delta$ and $\Gamma_1=\emptyset$ and that $\partial\Omega\cap\partial\Omega_+$ satisfies the geometrical condition (18) of [FLN], which is valid if all sectional curvatures at each point are strictly positive. Let $\mathcal{S}$ be a set of positive solutions of (1.1) such that $\Lambda_{\mathcal{S}}$ is bounded. Then step 2 of the proof of Theorem 1.1 in [FLN] implies uniform a priori bounds for $\mathcal{S}$ near $\partial\Omega\cap\partial\Omega_+$. Thus in this case condition (4.10) is not required on $\partial\Omega\cap\partial\Omega_+$. $\square$

5 A Priori Bounds By Harnack's Inequality. If we do not impose a decay condition on $a^+(x)$ as x approaches $\partial\Omega_+$, that is, if $\gamma=0$ in (4.10), then Theorem 4.3 guarantees a priori bounds for positive solutions of (1.1) if $r<(n+1)/(n-1)$. This restriction is also required in the main theorem of [BT]. Now we show that this bound can be improved, provided

$$\overline{\Omega}_+\subset\Omega \ , \quad \overline{\Omega}_+\cap\overline{\Omega}_-=\emptyset \ . \tag{5.1}$$

For this we first establish a preliminary estimate.

5.1 Lemma. *Let* (5.1) *hold and let $\mathcal{S}$ be a set of positive solutions of* (1.1) *such that $\Lambda_{\mathcal{S}}$ is bounded. Then there exists an open set Ω^* with $\Omega_+ \subset\subset \Omega^* \subset\subset \Omega \backslash \overline{\Omega}_-$ such that*

$$\sup_{(u,\lambda)\in\mathcal{S}} \|u\|_{L_p(\Omega^*)} < \infty \ , \tag{5.2}$$

provided $p \in [1,\infty)$ satisfies $p < n/(n-2)$ if $n \geq 3$.

Proof. Fix $R > 0$ such that $\overline{\Omega}_+ + \mathbb{B}_{2R} \subset\subset \Omega\backslash\overline{\Omega}_-$. Then

$$\mathcal{A}u = \lambda u + af(\cdot,u)u = \lambda u + a^+ f_+(\cdot,u)u \geq \lambda u \quad \text{in } x + \mathbb{B}_{2R}$$

for $x \in \overline{\Omega}_+$ and $(u,\lambda) \in \mathcal{S}$. Thus u is a positive supersolution of $\mathcal{A} - \lambda$ on $x + \mathbb{B}_{2R}$ for $x \in \overline{\Omega}_+$. Hence the weak Harnack inequality (e.g., [GT, Theorem 8.18]) implies the existence of a constant c such that

$$\|u\|_{L_p(x+\mathbb{B}_{2R})} \leq c\big(1 + \inf_{x+\mathbb{B}_R} u\big) \ , \qquad x \in \overline{\Omega}_+ \ , \quad (u,\lambda) \in \mathcal{S} \ . \tag{5.3}$$

Now we deduce from Theorem 3.3 and Lemma 3.4 (cf. the proof of Theorem 3.5) that, given any $x \in \overline{\Omega}_+$, the right-hand side of (5.2) is bounded above, uniformly for $(u,\lambda) \in \mathcal{S}$ and $x \in \overline{\Omega}_+$. Since $\overline{\Omega}_+$ is compact, there exist $x_0,\ldots,x_N \in \overline{\Omega}_+$ such that $\overline{\Omega}_+ \subset \bigcup_{j=0}^N (x_j + \mathbb{B}_{2R}) =: \Omega^*$. Consequently, (5.3) implies (5.2) and Ω^* has the asserted properties. $\square$

After these preparations we can prove the main result of this section.

5.2 Theorem. *Let* (5.1) *be satisfied and suppose that*

$$r < n/(n-2) \quad \text{if } n \geq 3 \ . \tag{5.4}$$

If $\mathcal{S}$ is a set of positive solutions of (1.1) *such that $\Lambda_{\mathcal{S}}$ is bounded then $\mathcal{S}$ is bounded in $C(\overline{\Omega}) \times \mathbb{R}$.*

Proof. Note that $\mathcal{S}$ is a set of positive solutions of $\big(\mathcal{A} - \lambda - af(\cdot,u)\big)u = 0$ in Ω^*. Fix a real number $p_0 > (r-1)n/2$ such that $p_0 < n/(n-2)$ if $n \geq 3$, which is possible thanks to (5.4). Hence it follows from (1.5) and the fact that $\Omega^* \subset \Omega\backslash\overline{\Omega}_-$, which implies $af(\cdot,u) = a^+ f_+(\cdot,u)$ on Ω^*, that condition (A3.3) of the appendix is satisfied, where $a_0(y,\eta,\lambda) := -\lambda - af(y,\eta)$ and $s := r-1$. Moreover, Lemma 5.1 guarantees that $\mathcal{S}$ is bounded in $L_{p_0}(\Omega^*) \times \mathbb{R}$. Thus we infer from Theorem A3.1 that $\mathcal{S}$ is bounded in $C(\overline{\Omega}_+) \times \mathbb{R}$. Now the assertion follows from Theorem 4.1. $\square$

If $n \geq 3$ then $(n+\gamma+1)/(n-1) < n/(n-2)$ iff $\gamma < 2/(n-2)$. Thus, given condition (5.1), Theorem 5.2 provides us with a priori bounds for a larger range of r-values than Theorem 4.3 if $\gamma < 2/(n-2)$ and $n \geq 3$.

6 Bounds For Radially Symmetric Solutions. In this section we show that we can obtain uniform a priori bounds for positive radially symmetric solutions under the sole assumption $r < (n+2)/(n-2)$ if $n \geq 3$, provided $\operatorname{supp}(a^+)$ is a ball.

6.1 Theorem. *Suppose that $0 < \rho < R < \infty$ and that $\Omega = \mathbb{B}_R$ and $\Omega_+ = \mathbb{B}_\rho$. Also suppose that $r < (n+2)/(n-2)$ if $n \geq 3$. Let $\mathcal{S}$ be a set of positive radially symmetric solutions of* (1.1) *such that $\Lambda_{\mathcal{S}}$ is bounded. Then $\mathcal{S}$ is bounded in $C(\overline{\Omega}) \times \mathbb{R}$.*

Proof. From Theorem 3.3 and Lemma 3.4 we deduce that $\sup_{(u,\lambda)\in\mathcal{S}} \inf_{\Omega_+} u$ is bounded above. Moreover, if $\omega > 0$ is sufficiently large,

$$(\mathcal{A} + \omega)u = (\lambda + \omega)u + a^+ f(\cdot,u)u > 0 \quad \text{in } \Omega_+$$

for $(u, \lambda) \in \mathcal{S}$. Hence we infer from Bony's maximum principle (cf. [B]) that each u attains its minimum over Ω_+ on $\partial\Omega_+$. Thus, since each u is radially symmetric and $\partial\Omega_+ = \partial\mathbb{B}_\rho$, we see that the family $\{ u \; ; \; (u, \lambda) \in \mathcal{S} \}$ is uniformly bounded on $\partial\Omega_+$. Consequently, this family is uniformly bounded on Ω_+ by Lemma 4.2. Now the assertion follows again from Theorem 4.1 □

6.2 Remarks. (a) The arguments of the preceding proof do not work if $\partial\Omega_+$ has at least two components since the solutions may blow up on one component and may still be uniformly bounded on another one.

(b) Very simple one-dimensional examples show that, in general, the conclusions of Theorem 1 in [GNN] might fail. In particular, $u(r)$ will not decrease with r, due to the variation of the coefficients. In fact, in higher-dimensional problems the symmetry of the positive solutions might be lost, as it occurs for the Laplacian on the annulus, for example. Theorem 6.1 provides us exclusively with a priori bounds for the radially symmetric solutions of (1.1). □

7 Existence And Multiplicity Results. In this section we denote by $\mathcal{S}_+$ the set of *all* positive solutions of (1.1) and assume that

$$\left.\begin{array}{l}\text{given any bounded interval } I,\\ \mathcal{U}_I := \{ u \; ; \; (u, \lambda) \in \mathcal{S}_+, \; \lambda \in I \} \text{ is bounded in } C(\overline{\Omega}).\end{array}\right\} \tag{7.1}$$

Note that, thanks to Theorems 4.3, 5.2, and 6.1, respectively, (7.1) is true if one of the following conditions is satisfied:

(i) the hypotheses of Theorem 4.3 are fulfilled;
(ii) $\overline{\Omega}_+ \subset \Omega$, $\overline{\Omega}_+ \cap \overline{\Omega}_- = \emptyset$, and $r < n/(n-2)$ if $n \geq 3$;
(iii) the assumptions of Theorem 6.1 are met and every positive solution of (1.1) is radially symmetric.

From (7.1) and standard elliptic theory we infer that

$$\left.\begin{array}{l}\text{given any bounded interval } I,\\ \mathcal{U}_I \text{ is bounded in } W_p^2(\Omega) \text{ and in } C^\alpha(\overline{\Omega})\\ \text{for each } p \in [1, \infty) \text{ and each } \alpha \in [0, 2).\end{array}\right\} \tag{7.2}$$

We put $I_0 := (\sigma^\Omega - 1, \sigma^D + 1)$ and $\beta > \sup\{ \|u\|_\infty \; ; \; u \in \mathcal{U}_{I_0} \}$ and fix

$$\omega > \sup\{ |\lambda| + \|af(\cdot, u)\|_\infty + \|a\partial f(\cdot, u)u\| \; ; \; \|u\|_\infty < \beta + 1, \; \lambda \in I_0 \} . \tag{7.3}$$

Then we denote by e the unique solution of

$$(\mathcal{A} + \omega)e = 1 \quad \text{in } \Omega , \qquad \mathcal{B}e = 0 \quad \text{on } \partial\Omega .$$

Note that e is strongly positive by Theorem 2.4. We write E for the Banach space consisting of all $u \in C(\overline{\Omega})$ for which there exists $\alpha = \alpha(u) > 0$ such that $-\alpha e < u < \alpha e$, endowed with the norm

$$u \mapsto \|u\| := \inf\{ \alpha > 0 \; ; \; -\alpha e < u < \alpha e \}$$

and the natural point-wise order. Then E is an ordered Banach space whose positive cone, P, is normal and has nonempty interior. Moreover, $E \hookrightarrow C(\overline{\Omega})$ (cf. [A2, Section 2]. Consequently,

$$X := \{\, u \in C(\overline{\Omega}) \ ;\ \|u\|_\infty < \beta \,\} \cap P$$

is a convex open subset of P containing 0.

It follows from Theorem 2.2 and (7.2) that $K := (\omega + A_p)^{-1} \,|\, E$ is well-defined and independent of $p > n$. Moreover, K is a compact endomorphism of E which is strongly positive, that is, $K\big(P \backslash \{0\}\big) \subset \mathring{P}$. We also put

$$F(u,\mu) := K\big((\omega + \sigma^\Omega + \mu)u + af(\cdot,u)u\big) \ , \qquad (u,\mu) \in E \times \mathbb{R} \ .$$

It is an easy result of our regularity assumptions on f and of $E \hookrightarrow C(\Omega) \hookrightarrow L_p(\Omega)$ that $F \in C^1(E \times \mathbb{R}, E)$ and that F is compact on bounded sets. From (7.3) and the strong positivity of K we infer that F maps $X \times (I_0 - \sigma^\Omega)$ into P and that $F(\cdot,\mu) \,|\, X \to P$ is strongly increasing for $\mu \in I_0 - \sigma^\Omega =: J$ (cf. [A2] for definitions and notations). Moreover, given $(u,\mu) \in X \times J$, the (right) derivative

$$F'(u,\mu) = \big(\partial_1 F(u,\mu), \partial_2 F(u,\mu)\big) : E \times \mathbb{R} \to E$$

of F is strongly positive.

In the following we denote by $r(u,\mu)$ the spectral radius of $\partial_1 F(u,\mu)$. Note that $\partial_1 F(0,\mu) = (\omega + \sigma^\Omega + \mu)K$. Since K is strongly increasing and compact it follows that $r(0,\mu)$ is positive if $\mu \in J$, that it is a simple eigenvalue of $\partial_1 F(0,\mu)$ possessing an eigenvector $\psi \in \mathring{P}$, and that $r(0,\mu)$ is the only eigenvalue of $\partial_1 F(0,\mu)$ having a positive eigenvector (cf [A2, Theorem 3.2]). Note that $\partial_1 F(0,\mu)\varphi = r(0,\mu)\varphi$ is equivalent to

$$\mathcal{A}\psi = \Big(\frac{\omega + \sigma^\Omega + \mu}{r(0,\mu)} - \omega\Big)\psi \quad \text{in } \Omega \ , \qquad \mathcal{B}\psi = 0 \quad \text{on } \partial\Omega \ .$$

Hence we infer from Theorem 2.2 that

$$r(0,\mu) = \frac{\omega + \sigma^\Omega + \mu}{\omega + \sigma^\Omega} \ , \qquad \mu \in J \ . \tag{7.4}$$

Now we put

$$\Sigma := \{\, (u,\mu) \in X \times J \ ;\ u = F(u,\mu),\ u \neq 0 \,\} \ .$$

Then

$$(u,\mu) \in \Sigma \quad \Longleftrightarrow \quad (u, \sigma^\Omega + \mu) \in \mathcal{S}_+ \text{ and } \mu \in J \ . \tag{7.5}$$

After these preparations we can prove the main results of this section. We begin by describing

$$\Lambda := \{\, \lambda \in \mathbb{R} \ ;\ \exists\ (u,\lambda) \in \mathcal{S}_+ \,\} \ ,$$

that is, the set of parameters λ for which $(3.4)_\lambda$ has a positive solution.

7.1 Theorem. *Either* $\Lambda = (-\infty, \sigma^\Omega)$ *or* $\Lambda = (-\infty, \lambda^*]$ *for some* $\lambda^* \in [\sigma^\Omega, \sigma^D)$.

Proof. From Rabinowitz' global bifurcation theorem [R] and the fact that σ^Ω is the only eigenvalue of $(\mathcal{A}, \mathcal{B}, \Omega)$ with a positive eigenfunction we infer that from the point $(0, \sigma^\Omega)$ of $C(\overline{\Omega}) \times \mathbb{R}$ there emanates a continuum $\mathcal{C}$ of positive solutions of (1.1), which is unbounded in $C(\overline{\Omega}) \times \mathbb{R}$. Hence (7.1) and Theorem 3.3 imply

$$(-\infty, \sigma^\Omega) \subset \Lambda \subset (-\infty, \sigma^D) \ .$$

Suppose that $\sigma^\Omega + \mu_1 \in \Lambda \cap (\sigma^\Omega, \sigma^D)$. Then there exists u_1 in $X \cap \mathring{P}$ such that $u_1 = F(u_1, \mu_1) \geq F(u_1, \mu)$ for $0 < \mu < \mu_1$. Moreover, $F(0, \mu) = 0$ and $r(0, \mu) > 1$, by (7.4). Hence [A2, Theorem 7.6] guarantees that $\sigma^\Omega + \mu$ belongs to Λ. Thus there exists $\mu^* \leq \sigma^D - \sigma^\Omega$ with $(\sigma^\Omega, \sigma^\Omega + \mu^*) \subset \Lambda$.

Let $\big((u_j, \mu_j)\big)_{j\in\mathbb{N}}$ be a sequence in Σ such that $\mu_j \to \mu_0$. Then the compactness of F implies that, by passing to a suitable subsequence, we may assume that $u_j \to u_0 \in X$. If $\mu_0 \neq 0$ it follows that $u_0 \in X \cap \mathring{P}$ since $(0, \sigma^\Omega) \in C(\overline{\Omega}) \times \mathbb{R}$ is the only bifurcation point of (1.1) from the line of trivial solutions from which emanates a branch of positive solutions. This implies, in particuler, that $\lambda^* := \sigma^\Omega + \mu^* \in \Lambda$.

If $\mu_0 = 0$ and $u_0 = 0$ then $(u_j, \sigma^\Omega + \mu_j) \in \mathcal{C}$ thanks to the fact that near $(0, \sigma^\Omega)$ in $C(\overline{\Omega}) \times \mathbb{R}$ all positive solutions of (1.1) are contained in $\mathcal{C}$ since we are dealing with bifurcation from a simple eigenvalue (cf. [CR]). This means that supercritical bifurcation occurs in this case. Since $\mathcal{S}_+ \cap \big(C(\overline{\Omega}) \times I_0\big)$ is contained in the bounded set $X \times I_0$ and Λ is bounded above by λ^*, we see that the global continuum $\mathcal{C}$ has to 'bend back'. This shows that there exists $u_0 \in \mathring{P}$ with $(u_0, \sigma^\Omega) \in \mathcal{S}_+$, that is, $\sigma^\Omega \in \Lambda$. This proves the theorem. □

The following proposition guarantees that $\lambda^* > \sigma^\Omega$ provided a^+ is sufficiently small.

7.2 Proposition. *Suppose that* $\sigma^\Omega < \eta < \sigma^D$. *Then there exists* $\varepsilon := \varepsilon(\eta) > 0$ *such that* $\lambda^* \geq \eta$ *provided* $\|a^+\|_\infty \leq \varepsilon$.

Proof. Define $G \in C^1\big(E \times L_\infty(\Omega_+), E\big)$ by

$$G(u, b) := u - K\big(\omega + \eta + bf_+(\cdot, u) - a^- f_-(\cdot, u)\big)u \ , \qquad (u, b) \in E \times L_\infty(\Omega_+) \ .$$

Then $G(\theta_\eta, 0) = 0$ (cf. Theorem 3.1 and observe that $\sigma^\Omega < \sigma^D$ implies $\Omega_- \neq \emptyset$). Moreover,

$$\partial_1 G(\theta_\eta, 0) = 1 - K\big(\omega + \eta - a^- f_-(\cdot, \theta_\eta) - a^- \partial f_-(\cdot, \theta_\eta)\theta_\eta\big) \ .$$

Hence $\partial_1 G(\theta_\eta, 0) \in \mathcal{L}(E)$ is a Fredholm operator of index zero and v belongs to its kernel iff

$$\begin{aligned} \big(\mathcal{A} - \eta + a^- f_-(\cdot, \theta_\eta) - a^- \partial f_-(\cdot, \theta_\eta)\theta_\eta\big)v &= 0 \qquad \text{in } \Omega \ , \\ \mathcal{B}v &= 0 \qquad \text{on } \partial\Omega \ . \end{aligned} \tag{7.6}$$

It follows from (1.4) that

$$\sigma^\Omega\big(\mathcal{A} - \eta + a^- f_-(\cdot, \theta_\eta) - a^- \partial f_-(\cdot, \theta_\eta)\theta_\eta, \mathcal{B}\big) > \sigma^\Omega\big(\mathcal{A} - \eta + a^- f_-(\cdot, \theta_\eta), \mathcal{B}\big) = 0 \ ,$$

where the last inequality is a consequence of $G(\theta_\eta, 0) = 0$ and Theorem 2.2. Hence we deduce from (7.6) that $v = 0$. Consequently, $\partial_1 G(\theta_\eta, 0)$ is an automorphism of E and the assertion follows from the implicit function theorem. □

The preceding proposition has the following counterpart which shows that $\lambda^* < \eta$ if a^+ is too large and f_+ is increasing in its last variable.

7.3 Proposition. *Suppose that there exists an open set Q of class C^2 with $\overline{Q} \subset \Omega_+$. Also suppose that $\sigma^\Omega < \eta < \sigma^D$ and*

$$\inf_Q a^+ > \frac{\sigma^Q(\mathcal{A}, \mathcal{B}_Q) - \eta}{\inf_Q f_+(\cdot, \theta_\eta)} \ . \tag{7.7}$$

The $\lambda^* < \eta$.

Proof. Let u_η be a positive solution of $(3.4)_\eta$. Then, by Lemma 3.4,

$$\inf_Q a^+ < \frac{\sigma^Q(\mathcal{A}, \mathcal{B}_Q) - \eta}{\inf_Q f(\cdot, u_\eta)} \ . \tag{7.8}$$

Moreover,

$$\mathcal{A} u_\eta = \eta u_\eta + a f(\cdot, u_\eta) u_\eta > \eta u_\eta - a^- f_-(\cdot, u_\eta) u_\eta \ .$$

Hence we infer from the strong maximum principle (cf. the last part of the proof of Theorem 4.1) that $u_\eta \ge \theta_\eta$. Thus we obtain from the fact that $f_+(x, \cdot)$ is increasing for $x \in Q$ and (7.8) a contradiction to (7.7). Now the assertion follows from Theorem 7.1. $\square$

Finally, we prove a multiplicity result in the case that $\lambda^* > \sigma^\Omega$.

7.4 Theorem. *Suppose that $\lambda^* > \sigma^\Omega$. Then $(3.4)_\lambda$ has for each $\lambda \in (\sigma^\Omega, \lambda^*)$ at least two positive solutions.*

Proof. Since we are interested in λ-values belonging to $[\sigma^\Omega, \sigma^D)$ only, thus to I_0, it follows from (7.5) that we can study the equivalent parameter-dependent fixed point equation $u = F(u, \mu)$ in $X \times J$. Hence it follows from the considerations at the beginning of this section that hypothesis (H) on page 680 of [A2] is satisfied (for F restricted to $X \times J$, which is all we need in the following). From [A2, Theorem 20.3] we know that $F(\cdot, \mu)$ possesses for $0 < \mu < \mu^*$ a least positive fixed point $\overline{u}(\mu)$ and that the map $\overline{u}(\cdot) : (0, \mu^*) \to \mathring{P}$ is strongly increasing and left continuous. Moreover, [A2, Proposition 20.4] guarantees that $r\big(\overline{u}(\mu), \mu\big) \le 1$ for $0 < \mu < \mu^*$. If $r\big(\overline{u}(\mu_0), \mu_0\big) < 1$ then $u_0 := \overline{u}(\mu_0)$ is an isolated fixed point of $F(\cdot, \mu_0)$ and the Leray-Schauder formula implies that the local fixed point index $i\big(F(\cdot, \mu_0), u_0\big)$ of $F(\cdot, \mu_0)$ at u_0 equals 1 (cf. [A2, Theorem 11.4]).

Suppose that $r(u_0, \mu_0) = 1$. Then there exist a neighborhood $V \times I$ of (u_0, μ_0) in $\mathring{P} \times J$, a positive number ε, and a continuously differentiable map $\big(u(\cdot), \mu(\cdot)\big)$ from $(-\varepsilon, \varepsilon)$ to $P \times \mathbb{R}$ such that $\big(u(0), \mu(0)\big) = (u_0, \mu_0)$ and

$$\Sigma \cap (V \times I) = \big\{ \big(u(t), \mu(t)\big) \ ; \ -\varepsilon < t < \varepsilon \big\} \ . \tag{7.9}$$

Moreover, $u(\cdot)$ is strongly increasing and

$$\operatorname{sign} \mu'(t) = \operatorname{sign}\big(1 - r(u(t), \mu(t))\big) \ , \qquad |t| < \varepsilon \ . \tag{7.10}$$

This follows from [A2, Proposition 20.8] by observing that the continuous differentiability of F suffices for its proof (also see the proof of Theorem 2.1 in [A1]).

First we observe that $\mu(t) < \mu_0$ for $-\varepsilon < t < 0$. Indeed, $u(t) < u_0$ for $-\varepsilon < t < 0$ since $u(\cdot)$ is strongly increasing. If $\mu(s) \ge \mu_0$ for some $s \in (-\varepsilon, 0)$ then

$$u(s) \ge \overline{u}\big(\mu(s)\big) \ge \overline{u}(\mu_0) = u_0$$

since $\overline{u}(\cdot)$ is also increasing. Hence $\mu(t) < \mu_0$ for $-\varepsilon < t < 0$.

Next suppose that u_0 is the only fixed point of $F(\cdot,\mu_0)$ in V and $\mu(t) > \mu_0$ for $0 < t < \varepsilon$. Then there exists $s \in (0,\varepsilon)$ such that $\mu'(s) > 0$, and (7.10) implies $r\big(u(s),\mu(s)\big) < 1$. Thus $u(s)$ is an isolated fixed point of $F\big(\cdot,\mu(s)\big)$ and

$$i\big(F(\cdot,\mu(s)),u(s)\big) = 1 \ . \tag{7.11}$$

Put $X_\rho := \{\, u \in X \ ; \ \|u\| < \rho \,\}$ for $\rho > 0$. Then the strong monotonicity of $u(\cdot)$, the monotonicity of the norm (which is a consequence of the normality of P), the fact that u_0 is the least fixed point of $F(\cdot,\mu_0)$, and that we can choose s arbitrarily closse to 0, hence $\mu(s)$ arbitrarily close to μ_0, imply the existence of $0 < \rho < \sigma$ such that $u(s)$ is the only fixed point of $F\big(\cdot,\mu(s)\big)$ in $X_\sigma \backslash \overline{X}_\rho$. From (7.9) we also infer the existence of $\overline{\mu} > \mu(s)$ such that

$$F(u,\mu) \neq u \ , \qquad (u,\mu) \in \big(\partial X_\rho \times \big[\mu(s),\overline{\mu}\big]\big) \cup \big(X_\rho \times \{\overline{\mu}\}\big) \ .$$

Hence the homotopy invariance of the fixed point index (see [A2, Section 11]) entails

$$i\big(F(\cdot,\mu(s)),X_\rho,\overline{X}\big) = i\big(F(\cdot,\overline{\mu}),X_\rho,\overline{X}\big) = 0 \ .$$

Thus, by (7.11) and the additivity property,

$$i\big(F(\cdot,\mu(s)),X_\sigma,\overline{X}\big) = i\big(F(\cdot,\mu(s)),u(s)\big) + i\big(F(\cdot,\mu(s)),X_\rho,\overline{X}\big) = 1 \ .$$

Finally, observe that $F(u,\mu) \neq u$ for $u \in \partial X_\sigma \times \big[\mu_0,\mu(s)\big]$ thanks to the fact that $u(\cdot)$ is strongly increasing. Thus, by using the homotopy invariance once more,

$$1 = i\big(F(\cdot,\mu(s)),X_\sigma,\overline{X}\big) = i\big(F(\cdot,\mu_0),X_\sigma,\overline{X}\big) = i\big(F(\cdot,\mu_0),u_0\big) \ ,$$

where the last equality sign is valid since u_0 is the only fixed point of $F(\cdot,\mu_0)$ in X_σ.

Note that $F(\cdot,\mu)$ has no fixed point in $\overline{X}$ for $\mu > \mu^*$. Hence, by homotopy invariance,

$$i\big(F(\cdot,\mu_0),X,\overline{X}\big) = i\big(F(\cdot,\mu),X,\overline{X}\big) = 0 \ .$$

Thus, by the additivity property, $F(\cdot,\mu_0)$ has at least two fixed points in X if u_0 is an isolated fixed point of $F(\cdot,\mu_0)$ with local index 1. By the above considerations this is the case if either $r(u_0,\mu_0) < 1$ or $r(u_0,\mu_0) = 1$ with u_0 being the only fixed point of $F(\cdot,\mu_0)$ in V and $\mu(t) > \mu_0$ for $0 < t < \varepsilon$.

It remains to consider the case where u_0 is the only fixed point of $F(\cdot,\mu_0)$ in V and $r(u_0,\mu_0) = 1$ as well as $\mu(t) < \mu_0$ for $0 < t < \varepsilon$. Since $\mu_0 < \mu^*$ it follows that $\overline{u}(\mu)$, the least fixed point of $F(\cdot,\mu)$, belongs to $X \backslash \overline{V}$ for $\mu_0 < \mu \leq \mu^*$. Note that

$$Y := \big\{\, \overline{u}(\mu) \ ; \ \mu_0 < \mu \leq \mu^* \,\big\} \subset F\big(X_\rho \times (\mu,\mu^*]\big) \ ,$$

where $\rho := \|\overline{u}(\mu^*)\|$. Hence Y is a relatively compact subset of $X \backslash \overline{V}$ thanks to the compactness of F on bounded sets. Thus there exist a sequence (μ_j) in $(\mu_0,\mu^*]$ converging towards μ_0 and $v \in X \backslash \overline{V}$ with $\overline{u}(\mu_j) \to v$. Hence $(v,\mu_0) \in \Sigma$ and $v \neq u_0$ which show that $F(\cdot,\mu_0)$ possesses two fixed points in this case as well. $\square$

It should be remarked that the above proof is an elaboration of the remarks following the proof of Theorem 20.8 in [A2].

Appendix: Interior L_p-Estimates. In this appendix we derive interior elliptic L_p-estimates, where we impose minimal smoothness hypotheses for the coefficients. Then we show how these results can be combined with bootstrapping arguments to improve given a priori bounds.

It turns out that it is only slightly more difficult to consider rather general elliptic systems of arbitrary order than to treat the case of a single second order equation. For this reason — and for further use — we deal with the general situation.

In principle, the results of this appendix are known to specialists in the theory of partial differential equations, and the techniques which we use are well-known (cf. [M], [H]). However, we believe that our main result, namely Theorem A2.1, is new as far as the minimal smoothness assumptions for the coefficents are concerned. In any case, we could not find a precise statement of the needed a priori estimates in the literature so that we decided to include proofs.

A1 Preliminaries. Let F be a finite-dimensional Banach space over $\mathbb{R}$ or $\mathbb{C}$, and suppose that $1 < p < \infty$.

We denote by $H_p^s := H_p^s(\mathbb{R}^n, F)$ the Bessel potential space of order s, and we write $\|\cdot\|_{s,p}$ for its norm. Recall that $H_p^k = W_p^k$ for $k \in \mathbb{N}$, except for equivalent norms. Also recall that $H_p^s \hookrightarrow H_p^t$ for $s > t$ and that the (generalized) Sobolev embedding theorem asserts that

$$H_p^s \hookrightarrow H_q^t\ , \qquad s - n/p = t - n/q\ , \quad 1/p \geq 1/q > 0\ , \tag{A1.1}$$

and

$$H_p^s \hookrightarrow W_\infty^k\ , \qquad s - n/p > k\ , \tag{A1.2}$$

where $k \in \mathbb{N}$. Moreover, given $0 < s < t < \infty$,

$$\|u\|_{s,p} \leq c\,\|u\|_p^{1-s/t}\,\|u\|_{t,p}^{s/t}\ , \qquad u \in H_p^t\ . \tag{A1.3}$$

For proofs and more details on these spaces we refer to [T, Chapter II] and [A4, Chapter VII], for example.

Given $k \in \mathbb{N}$ and $s \geq k$, we define $q_k(s,p) \in [p,\infty]$ by

$$q_k(s,p) := \begin{cases} n/(s-k) & \text{if } s-k < n/p\ , \\ > p & \text{if } s-k = n/p\ , \\ p & \text{if } s-k > n/p\ . \end{cases} \tag{A1.4}$$

Then it follows from (A1.1) and (A1.2) that

$$H_p^s \hookrightarrow W_{r_k(s,p)}^k\ , \qquad \frac{1}{r_k(s,p)} := \frac{1}{p} - \frac{1}{q_k(s,p)}\ . \tag{A1.5}$$

Using these facts we can now prove the following interpolation-type estimate, where $m \in \mathbb{N}$.

A1.1 Lemma. *Let $\alpha \in \mathbb{N}^n$ satisfy $|\alpha| \le k \le m$ and suppose $q_k(m,p) \le q < \infty$. Also suppose that $\mathfrak{A}$ is a bounded subset of $L_q\big(\mathbb{R}^n, \mathcal{L}(F)\big)$ and that one of the following conditions is satisfied:*

(i) *$m-k \ge n/p$ and $q = q_k(m,p)$;*
(ii) *$m-k < n/p$ and either $|\alpha| < k$ or $q > q_k(m,p)$;*
(iii) *$m-k < n/p$ and $\mathfrak{A}$ is compact.*

Then there exists for each $\varepsilon > 0$ a constant $c(\varepsilon)$ such that

$$\|a\partial^\alpha u\|_p \le \varepsilon \|u\|_{m,p} + c(\varepsilon) \|u\|_p \ , \qquad a \in \mathfrak{A} \ , \quad u \in W_p^k \ .$$

Proof. Define $r \in [p,\infty]$ by $r^{-1} = p^{-1} - q^{-1}$. Then Hölder's inequality implies

$$\|a\partial^\alpha u\|_p \le \|a\|_q \|\partial^\alpha u\|_r \ . \tag{A1.6}$$

Next we derive estimates for $\|\partial^\alpha u\|_r$, given either one of conditions (i) and (ii).

(i) If $m-k > n/p$ then $q = p$ and $r = \infty$. Fix $s \in (k,m)$ with $s-k > n/p$. Then we infer from (A1.2) that

$$\|\partial^\alpha u\|_r \le \|u\|_{k,r} \le c \|u\|_{s,p} \ . \tag{A1.7}$$

If $m-k = n/p$ then $q > p$. We put $s := |\alpha| + n/q$ and observe that

$$|\alpha| < s = |\alpha| + n/q < |\alpha| + n/p \le k + n/p = m \ .$$

Since $s - |\alpha| = n(p^{-1} - r^{-1})$, we see that $r = r_{|\alpha|}(s,p)$. Hence it follows from (A1.5) that estimate (A1.7) is true.

(ii) Here we also put $s := |\alpha| + n/q$. Then

$$|\alpha| < s = |\alpha| + n/q < k + n/q_k(m,p) = m$$

and $s - |\alpha| = n/q < n/p$, since $q > p$. Thus we find again that r equals $r_{|\alpha|}(s,p)$, so that (A1.5) implies estimate (A1.7) once more.

Now we derive from (A1.3), (A1.6), and (A1.7) that

$$\|a\partial^\alpha u\|_p \le c \|a\|_q \|u\|_p^{1-s/m} \|u\|_{m,p}^{s/m} \ , \qquad a \in \mathfrak{A} \ , \quad u \in W_p^m \ .$$

Hence the assertion follows in cases (i) and (ii) by a standard application of Young's inequality.

Now suppose that (iii) is true. Thanks to (ii) we can assume that $|\alpha| = k$ and $q = q_k(m,p)$. Consequently $r = r_k(m,p)$, and (A1.5) implies

$$\|\partial^\alpha u\|_r \le \|u\|_{k,r} \le c_0 \|u\|_{m,p} \ , \qquad u \in W_p^m \ . \tag{A1.8}$$

Let $\varepsilon > 0$ be given. Since $\mathfrak{A}$ is compact, there exist $a_1, \ldots, a_{N(\varepsilon)} \in \mathfrak{A}$ such that, for each $a \in \mathfrak{A}$, we find $j_a \in \big\{1, \ldots, N(\varepsilon)\big\}$ with

$$\|a - a_{j_a}\|_q < \varepsilon/(3c_0) \ . \tag{A1.9}$$

Recall that $\mathcal{D}$, the space of test functions, is dense in L_q. Hence for each j there exists $b_j \in \mathcal{D}$ with

$$\|a_j - b_j\|_q < \varepsilon/(3c_0) \ , \qquad 1 \le j \le N(\varepsilon) \ . \tag{A1.10}$$

Thus we deduce from $a = (a - a_{j_a}) + (a_{j_a} - b_{j_a}) + b_{j_a}$ and from (A1.6) and (A1.8)–(A1.10) that

$$\|a\partial^\alpha u\|_p \le (2\varepsilon/3)\, \|u\|_{m,p} + \max_{1\le j\le N(\varepsilon)} \|b_j\partial^\alpha u\|_p \tag{A1.11}$$

for $a \in \mathfrak{A}$ and $u \in W_p^m$. Note that $\{\, b_j \ ; \ 1 \le j \le N(\varepsilon)\,\}$ is a bounded subset of $L_{\widetilde{q}}\big(\mathbb{R}^n, \mathcal{L}(F)\big)$ for each $\widetilde{q} \in \big(q_k(m,p), \infty\big]$. Hence, by applying the already proven estimate for case (ii), we see that

$$\max_{1\le j\le N(\varepsilon)} \|b_j\partial^\alpha u\|_p \le (\varepsilon/3)\, \|u\|_{m,p} + c(\varepsilon)\, \|u\|_p \ , \qquad u \in W_p^m \ ,$$

which, together with (A1.11), proves the assertion in this situation as well. □

A1.2 Remark. Suppose that there exists a bounded open subset Y of $\mathbb{R}^n$ such that $\operatorname{supp}(a) \subset Y$ for each $a \in \mathfrak{A}$. Then Lemma A1.1 remains valid if $q = \infty$.

Proof. Fix $\chi \in \mathcal{D}$ such that $\chi\,|\,Y = 1$. Then, by identifying a with χa, it follows that $\mathfrak{A}$ is a bounded subset of $L_{\widetilde{q}}\big(\mathbb{R}^n, \mathcal{L}(F)\big)$ for each $\widetilde{q} \in [1,\infty)$. Thus we can replace $q = \infty$ by a suitable $q \in (p, \infty)$. □

A2 Interior Estimates. Put $D_j := -i\partial_j$ for $1 \le j \le n$ and let Y be a bounded open subset of $\mathbb{R}^n$. Suppose that $E_\alpha(Y)$ is for each $|\alpha| \le k$ a Banach space of $\mathcal{L}(F)$-valued functions on Y. Then we denote by $\operatorname{Diff}_k\big(\mathbb{E}(Y)\big)$ the set of all linear differential operators,

$$\mathcal{A} := \sum_{|\alpha|\le k} a_\alpha D^\alpha \ ,$$

of order at most k with the coefficients $(a_\alpha)_{|\alpha|\le k}$ belonging to

$$\mathbb{E}(Y) := \prod_{|\alpha|\le k} E_\alpha(Y) \ .$$

We topologize $\operatorname{Diff}_k\big(\mathbb{E}(Y)\big)$ by means of the identification

$$\operatorname{Diff}_k\big(\mathbb{E}(Y)\big) \ni \mathcal{A} = \sum_{|\alpha|\le k} a_\alpha D^\alpha \quad \longleftrightarrow \quad (a_\alpha)_{|\alpha|\le k} \in \mathbb{E}(Y) \ ,$$

which identifies $\operatorname{Diff}\big(\mathbb{E}(Y)\big)$ with $\mathbb{E}(Y)$. We write $\pi\mathcal{A}$ for the principal part of $\mathcal{A}$, given by $\sum_{|\alpha|=k} a_\alpha D^\alpha$, and $\pi\mathcal{A}(y)$ for the homogeneous differential operator of order k with constant coefficients obtained from $\pi\mathcal{A}$ by freezing the coefficients at $y \in Y$. Moreover, the principal symbol of $\mathcal{A}$ is defined by

$$\pi\mathcal{A}(y,\xi) := \sum_{|\alpha|=k} a_\alpha(y)\xi^\alpha \ , \qquad y \in Y \ , \quad \xi \in \mathbb{R}^n \ .$$

Then $\mathcal{A}$ is said to be uniformly regularly elliptic if there exists an 'ellipticity constant' $\overline{\varepsilon} > 0$ and an 'angle of ellipticity' $\overline{\theta} \in [0, \pi)$ such that

$$\operatorname{spec}\bigl(\pi\mathcal{A}(y,\xi)\bigr) \subset \bigl\{\, z \in \mathbb{C} \; ; \; |z| \geq \overline{\varepsilon}, \; |\arg z| \leq \overline{\theta} \,\bigr\} , \qquad y \in Y , \quad \xi \in S^{n-1} ,$$

where spec$(\dots)$ denotes the spectrum.

Fix $m \in \mathbb{N}\backslash\{0\}$. For $1 < p < \infty$ we put

$$\mathbb{E}_p(Y) := \prod_{|\alpha|=m} C\bigl(\overline{Y}, \mathcal{L}(F)\bigr) \times \prod_{|\alpha|<m} L_{q_{|\alpha|}(m,p)}\bigl(Y, \mathcal{L}(F)\bigr)$$

and

$$\mathcal{E}\ell\ell_p(Y) := \bigl\{\, \mathcal{A} \in \mathbb{E}_p(Y) \; ; \; \mathcal{A} \text{ is uniformly regularly elliptic} \,\bigr\} .$$

It is an easy consequence of the upper semicontinuity of the spectrum that $\mathcal{E}\ell\ell_p(Y)$ is open in $\mathbb{E}_p(Y)$.

For each $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m-1$ we fix $q_\alpha \in \bigl[q_{|\alpha|}(m,p), \infty\bigr]$ such that

$$q_\alpha > q_{|\alpha|}(m,p) = n/(m-|\alpha|) \qquad \text{if} \quad m - |\alpha| < n/p .$$

Then we put

$$\mathbb{E}_{p,\vec{q}}(Y) := \prod_{|\alpha|\leq m-1} L_{q_\alpha}\bigl(Y, \mathcal{L}(F)\bigr) .$$

Using these notations we can prove the following general interior L_p-estimates for elliptic systems.

A2.1 Theorem. *Suppose that $\mathfrak{A}$ is a compact subset of $\mathcal{E}\ell\ell_p(Y)$, and $\mathfrak{B}$ is a bounded subset of $\mathbb{E}_{p,\vec{q}}(Y)$. Then, given any open subset $X \subset\subset Y$, there exists a constant c such that*

$$\|u\|_{m,p,X} \leq c\bigl(\|(\mathcal{A}+\mathcal{B})u\|_{p,Y} + \|u\|_{p,Y}\bigr)$$

for all $u \in W_p^m(Y,F)$, $\mathcal{A} \in \mathfrak{A}$, and $\mathcal{B} \in \mathfrak{B}$.

Proof. By means of the upper semicontinuity of the spectrum and the compactness of $\mathfrak{A}$ it is not difficult to see that there exist $\overline{\varepsilon} > 0$ and $\overline{\theta} \in (0, \pi)$ such that each $\mathcal{A} \in \mathfrak{A}$ is uniformly regularly elliptic with ellipticity constant $\overline{\varepsilon}$ and angle of ellipticity $\overline{\theta}$. Moreover, if a_α are the coefficients of $\mathcal{A}$ then

$$\sup_{\mathcal{A}\in\mathfrak{A}} \max_{|\alpha|=m} \|a_\alpha\|_\infty < \infty .$$

These facts and Mikhlin's multiplier theorem imply that $u \mapsto \bigl(1 + \pi\mathcal{A}(y)\bigr)u$ is a topological linear isomorphism from $W_p^m := W_p^m(\mathbb{R}^n, F)$ onto $L_p := L_p(\mathbb{R}^n, F)$ whose inverse is uniformly bounded with respect to $y \in Y$ and $\mathcal{A} \in \mathfrak{A}$ (e.g., Lemma 7.2 of [AHS] or [A4, Subsections VII.2.3 and VII.2.4]). Hence there exists $\kappa \geq 0$ with

$$\|u\|_{m,p} \leq \kappa\bigl(\|\pi\mathcal{A}(y)u\|_p + \|u\|_p\bigr) , \qquad u \in W_p^m , \quad \mathcal{A} \in \mathfrak{A} , \quad y \in Y . \tag{A2.1}$$

Denote by σ_t dilation for $t > 0$, defined by $\sigma_t u(x) := u(tx)$, and observe that

$$\partial^\alpha \circ \sigma_t = t^{|\alpha|}\sigma_t \circ \partial^\alpha \quad \text{and} \quad \|\sigma_t u\|_p = t^{-n/p}\,\|u\|_p .$$

Then, by replacing u in (A2.1) by $\sigma_t u$, we see that

$$\sum_{|\alpha|\le m} t^{|\alpha|}\, \|\partial^\alpha u\|_p \le \kappa\bigl(t^m\, \|\pi\mathcal{A}(y)u\|_p + \|u\|_p\bigr) \tag{A2.2}$$

for all $\mathcal{A}\in\mathfrak{A}$, $u\in W_p^m$, $y\in Y$, and $t>0$.

We write $Q:=(-1,1)^n$ for the open unit-ball of $\mathbb{R}^n$ with respect to the maximum norm. We also fix $\varphi_1,\varphi_2\in\mathcal{D}(2Q)$ such that $0\le\varphi_j\le 1$ and $\varphi_1\,|\,Q=1$ and $\varphi_2\,|\,\mathrm{supp}(\varphi_1)=1$. Lastly, $Q(x,r):=x+rQ$ for $x\in\mathbb{R}^n$ and $r>0$.

Let $\varepsilon>0$ be given. Since $\mathfrak{A}$ is compact in $\mathcal{E}\ell\ell_p(Y)$, the set $\pi\mathfrak{A}:=\{\,\pi\mathcal{A}\ ;\ \mathcal{A}\in\mathfrak{A}\,\}$ is compact in $E:=C\bigl(\overline{Y},\mathcal{L}(F)\bigr)^{M(m)}$, where $M(m)$ is the number of multiindices of length m. Hence there exist $\mathcal{A}_1,\dots,\mathcal{A}_{N(\varepsilon)}$ in $\mathfrak{A}$ such that we find for each $\mathcal{A}\in\mathfrak{A}$ an index $j_{\mathcal{A}}\in\bigl\{1,\dots,N(\varepsilon)\bigr\}$ with

$$\max_{|\alpha|=m}\|a_\alpha - a_{j_{\mathcal{A}},\alpha}\|_\infty \le \varepsilon\ ,$$

where $a_{j_{\mathcal{A}},\alpha}$ are the coefficients of $\mathcal{A}_{j_{\mathcal{A}}}$.

Note that, given $y\in Y$, and denoting by b_α the coefficients of $\mathcal{B}$,

$$\begin{aligned}\pi\mathcal{A}_{j_{\mathcal{A}}}(y) &= (\mathcal{A}+\mathcal{B}) + (\pi\mathcal{A}-\mathcal{A}-\mathcal{B}) + (\pi\mathcal{A}_{j_{\mathcal{A}}}-\pi\mathcal{A}) + \bigl(\pi\mathcal{A}_{j_{\mathcal{A}}}(y)-\pi\mathcal{A}_{j_{\mathcal{A}}}\bigr)\\ &= (\mathcal{A}+\mathcal{B}) - \sum_{|\alpha|\le m-1}(a_\alpha+b_\alpha)D^\alpha\\ &\qquad + \sum_{|\alpha|=m}\bigl[(a_{j_{\mathcal{A}},\alpha}-a_\alpha) + \bigl(a_{j_{\mathcal{A}},\alpha}(y)-a_{j_{\mathcal{A}},\alpha}\bigr)\bigr]D^\alpha\ .\end{aligned} \tag{A2.3}$$

Fix $r\in(0,1)$ and $y\in Y$ with $Q_{2r}:=Q(y,2r)\subset Y$. Also put

$$\psi_j(x):=\varphi_j\bigl((x-y)/r\bigr)\ ,\qquad x\in\mathbb{R}^n\ .$$

Then we infer from (A2.3), Lemma A1.1, and Remark A1.2 that

$$\|\pi\mathcal{A}_{j_{\mathcal{A}}}(y)(\psi_1 u)\|_p \le \|(\mathcal{A}+\mathcal{B})(\psi_1 u)\|_p + \bigl(2\varepsilon+\rho(r)\bigr)\,\|\psi_1 u\|_{m,p} + c(\varepsilon)\,\|\psi_1 u\|_p\ , \tag{A2.4}$$

where

$$\rho(r):=\max_{1\le j\le N(\varepsilon)}\ \max_{|\alpha|=m}\|a_{j_{\mathcal{A}},\alpha}-a_{j_{\mathcal{A}},\alpha}(y)\|_{\infty,Q(y,2r)}\ .$$

Since $\pi\mathfrak{A}$ is compact in E, this set is uniformly equicontinuous. Hence ρ is an increasing function of r such that $\rho(r)\to 0$ as $r\to 0$, independently of $y\in Y$ and $\varepsilon>0$.

Note that $\partial^\alpha\psi_j(x)=r^{-|\alpha|}\,\partial^\alpha\varphi_j\bigl((x-y)/r\bigr)$ for $j=1,2$. Hence, by Leibniz' rule,

$$\|\partial^\alpha(\psi_j u)\|_p \le c\sum_{\beta\le\alpha} r^{|\beta|-|\alpha|}\,\|\partial^\beta u\|_{p,Q_{2r}}\ . \tag{A2.5}$$

Consequently,

$$r^m\,\|\psi_1 u\|_{m,p} \le c\sum_{|\alpha|\le m} r^{|\alpha|}\,\|\partial^\alpha u\|_{p,Q_{2r}}\ . \tag{A2.6}$$

From Leibniz' rule we also deduce that

$$r^m \left\|(\mathcal{A}+\mathcal{B})(\psi_1 u)\right\|_p \le r^m \left\|\psi_1(\mathcal{A}+\mathcal{B})u\right\|_p + R\ , \tag{A2.7}$$

where

$$\begin{aligned} R := c \sum_{|\alpha|=m} \sum_{\beta<\alpha} r^{|\beta|} \, \|\partial^\beta u\|_{p,Q_{2r}} \\ + c \sum_{|\alpha|\le m-1} \sum_{\beta<\alpha} r^{m-|\alpha|+|\beta|} \big(\|a_\alpha \partial^\beta(\psi_2 u)\|_p + \|b_\alpha \partial^\beta(\psi_2 u)\|_p \big)\ . \end{aligned}$$

Note that $q_{|\alpha|}(m,p) = q_{|\beta|}(m-|\alpha|+|\beta|,p)$. Hence it follows from (A1.6), (A1.5), and the boundedness of $\mathfrak{A}$ that

$$\|a_\alpha \partial^\beta(\psi_2 u)\|_p \le c\, \|\psi_2 u\|_{|\beta|, r_{|\alpha|}(m,p)} \le c\, \|\psi_2 u\|_{m-|\alpha|+|\beta|,p}\ .$$

Consequently,

$$\begin{aligned} r^{m-|\alpha|+|\beta|} \, \|a_\alpha \partial^\beta(\psi_2 u)\|_p &\le c \sum_{|\gamma|\le m-|\alpha|+|\beta|} r^{m-|\alpha|+|\beta|} \, \|\partial^\gamma(\psi_2 u)\|_p \\ &\le c \sum_{|\alpha|\le m-1} r^{|\alpha|} \, \|\partial^\alpha u\|_{p,Q_{2r}}\ , \end{aligned}$$

where we used (A2.5) once more. Since Y is bounded it follows that

$$b_\alpha \in L_{q_{|\alpha|}(m,p)}\big(Y, \mathcal{L}(F)\big)\ , \qquad |\alpha| \le m-1\ .$$

Hence the preceding arguments show that

$$r^{m-|\alpha|+|\beta|} \, \|b_\alpha \partial^\beta(\psi_2 u)\|_p \le c \sum_{|\alpha|\le m-1} r^{|\alpha|} \, \|\partial^\alpha u\|_{p,Q_{2r}}\ ,$$

so that

$$R \le c \sum_{|\alpha|\le m-1} r^{|\alpha|} \, \|\partial^\alpha u\|_{p,Q_{2r}}\ . \tag{A2.8}$$

Now we replace t in (A2.2) by r/τ for $r,\tau>0$ and obtain, by using (A2.4) and (A2.6)–(A2.8), that

$$\begin{aligned} &\sum_{|\alpha|\le m} \tau^{(m-|\alpha|)p} r^{|\alpha|p} \, \|\partial^\alpha u\|^p_{p,Q(y,r)} \\ &\quad\le c r^{mp} \, \|(\mathcal{A}+\mathcal{B})u\|^p_{p,Q(y,2r)} + c\big(\varepsilon+\rho(r)\big)^p \sum_{|\alpha|=m} r^{mp} \, \|\partial^\alpha u\|^p_{p,Q(y,2r)} \\ &\qquad + c \sum_{|\alpha|\le m-1} r^{|\alpha|p} \, \|\partial^\alpha u\|^p_{p,Q(y,2r)} + \tau^{mp} c(\varepsilon) \, \|u\|^p_{p,Q(y,2r)} \end{aligned} \tag{A2.9}$$

for $u \in W_p^m(X,F)$ and $y \in Y$ with $Q(y,2r) \subset Y$, $0<\varepsilon\le 1$, and $(\mathcal{A},\mathcal{B}) \in \mathfrak{A}\times\mathfrak{B}$.

Observe that $\{ Q(rz,r) \ ; \ z \in \mathbb{Z}^n \}$ is for each $r>0$ an open covering of $\mathbb{R}^n$ such that each point of $\mathbb{R}^n$ is contained in at most 2^n of these sets. Given any nonempty subset Z of $\mathbb{R}^n$, we put

$$Z(r) := \{ x \in \mathbb{R}^n \ ; \ \mathrm{dist}_\infty(x,Z) < r \} \ , \qquad r > 0 \ ,$$

where dist_∞ denotes the distance with respect to the maximum norm. It is easily seen that

$$Z(r)(s) = Z(r+s) \ , \qquad r,s > 0 \ . \tag{A2.10}$$

Using these facts we infer from (A2.9) that

$$\begin{aligned} &\sum_{|\alpha|\le m} \tau^{(m-|\alpha|)p} r^{|\alpha|p} \, \|\partial^\alpha u\|_{p,Z}^p \\ &\qquad \le c r^{mp} \, \|(\mathcal{A}+\mathcal{B})u\|_{p,Z(r)}^p + c\bigl(\varepsilon + \rho(r)\bigr)^p \sum_{|\alpha|=m} r^{mp} \, \|\partial^\alpha u\|_{p,Z(r)}^p \\ &\qquad\quad + c \sum_{|\alpha|\le m-1} r^{|\alpha|p} \, \|\partial^\alpha u\|_{p,Z(r)}^p + \tau^{mp} c(\varepsilon) \, \|u\|_{p,Z(r)}^p \end{aligned} \tag{A2.11}$$

for any measurable nonempty subset Z of Y and any $r \in (0,1)$ with $Z(r) \subset Y$.

Now suppose that $r \in (0,1)$ is so small that

$$Y_0 := \{ y \in Y \ ; \ \mathrm{dist}_\infty(y, Y^c) > r \} \ne \emptyset \ .$$

Put $Y_{j+1} := Y_j(2^{-j-1}r) \setminus \bigcup_{k=0}^j Y_k$ for $j \in \mathbb{N}$. Then we infer from (A2.10) that

$$\bigcup_{j=0}^k Y_j = Y_0\bigl(\sum_{i=1}^k 2^{-i} r\bigr) \ ,$$

where the empty has the value 0, and $Y_0(0) := Y_0$. Thus $\bigcup_{j=0}^\infty Y_j = Y_0(r) = Y$ and

$$Y_{j+1} \subset Y_0\Bigl(\sum_{i=1}^{j+1} 2^{-i} r\Bigr) \Big\backslash Y_0\Bigl(\sum_{i=1}^{j} 2^{-i} r\Bigr) \ , \qquad j \ge 0 \ . \tag{A2.12}$$

Set $d(y) := \min\bigl\{1, \mathrm{dist}_\infty(y, Y^c)\bigr\}$. Then it follows from (A2.12) that

$$\frac{r}{2^j} \le d(y) \le \frac{r}{2^{j-1}} \ , \qquad y \in Y_j \ , \quad j \ge 1 \ . \tag{A2.13}$$

Also note that $Y_j(2^{-j-1}r) \subset Y_{j-1} \cup Y_j \cup Y_{j+1}$ for $j \ge 1$. Using these facts we easily deduce from (A2.13), by putting $Z := Y_j$ and replacing r by $2^{-j-1}r$, and by setting $Z_j := Y_{j-1} \cup Y_j \cup Y_{j+1}$ for $j \ge 1$ and $Z_0 := Y_0 \cup Y_1$, that

$$\begin{aligned} &\sum_{|\alpha|\le m} \tau^{(m-|\alpha|)p} \, \|d^{|\alpha|}\partial^\alpha u\|_{p,Y_j}^p \\ &\qquad \le c \, \|d^m(\mathcal{A}+\mathcal{B})u\|_{p,Z_j}^p + c\bigl(\varepsilon + \rho(r)\bigr)^p \sum_{|\alpha|=m} \|d^m \partial^\alpha u\|_{p,Z_j}^p \\ &\qquad\quad + c \sum_{|\alpha|\le m-1} \|d^{|\alpha|}\partial^\alpha u\|_{p,Z_j}^p + \tau^{mp} c(\varepsilon) \, \|u\|_{p,Z_j}^p \end{aligned}$$

for $j \in \mathbb{N}$. After summing these inequalities we arrive at

$$\begin{aligned}\sum_{|\alpha|\le m} \tau^{m-|\alpha|} \, \|d^{|\alpha|}\partial^\alpha u\|_{p,Y} \\ &\le c\,\|d^m(\mathcal{A}+\mathcal{B})u\|_{p,Y} + c\big(\varepsilon+\rho(r)\big)\sum_{|\alpha|=m}\|d^m\partial^\alpha u\|_{p,Y} \\ &\quad + c\sum_{|\alpha|\le m-1}\|d^{|\alpha|}\partial^\alpha u\|_{p,Y} + \tau^m c(\varepsilon)\,\|u\|_{p,Y} \ .\end{aligned}$$

By fixing a sufficiently large value of τ we can cancel the second to the last term against one half of the left-hand side. Then we fix ε and r so small that we can cancel the second term on the right against one half of the left-hand side. These operations lead to

$$\sum_{|\alpha|\le m}\|d^{|\alpha|}\partial^\alpha u\|_{p,Y} \le c\big(\|d^m(\mathcal{A}+\mathcal{B})u\|_{p,Y} + \|u\|_{p,Y}\big)$$

for $u \in W_p^m(Y,F)$ and $(\mathcal{A},\mathcal{B}) \in \mathfrak{A}\times\mathfrak{B}$. Now the assertion is obvious, since X is at a positive distance from Y^c. $\square$

A2.2 Remark. It is worthwhile to point out the particular case of a single operator, that is, $\mathfrak{A} = \{\mathcal{A}\}$ with $\mathcal{A} \in \mathcal{E}\ell\ell_p(Y)$ and $\mathfrak{B} = \emptyset$. Then it is obvious that the lower order coefficients of $\mathcal{A}$ satisfy the minimal smoothness assumptions that are needed to guarantee that $\mathcal{A} \in \mathcal{L}\big(W_p^m(Y,F), L_p(Y,F)\big)$, provided $m - n/p$ is not one of the integers $0, 1, \ldots, m-1$. In the particularly important second order case we see, for example, that $a_\alpha \in L_n\big(Y, \mathcal{L}(F)\big)$ for $|\alpha| = 1$ and $a_0 \in L_{n/2}\big(Y, \mathcal{L}(F)\big)$, provided $1 < p < n/2$. $\square$

A3 Bootstrapping. For simplicity — and in view of what is needed in this paper — we now restrict ourselves to the second order case and leave it to the reader to consider systems of arbitrary order.

We assume that Λ is a compact metric space and that

$$a_{j,k} \in C\big(\overline{Y}\times\Lambda, \mathcal{L}(F)\big) \ , \qquad 1 \le j,k \le n \ ,$$

such that

$$\operatorname{spec}\Big(\sum_{j,k=1}^n a_{j,k}(y,\lambda)\xi_j\xi_k\Big) \subset \mathbb{C}\backslash\mathbb{R}^+ \ , \qquad (y,\lambda,\xi) \in \overline{Y}\times\Lambda\times S^{n-1} \ , \tag{A3.1}$$

where $n \ge 2$. We also suppose that

$$a_j, a_0 \in C\big(\overline{Y}\times F\times\Lambda, \mathcal{L}(F)\big) \ , \quad 1 \le j \le n \ , \qquad f \in C(\overline{Y}\times\Lambda, F) \ ,$$

though weaker assumptions concerning the dependence on $y \in Y$ would suffice for what follows. Lastly, we assume that there exists $s \in [1,\infty)$ such that

$$|a_j(y,\eta,\lambda)| \le c(1+|\eta|^{s/2}) \ , \qquad 1 \le j \le n \ , \tag{A3.2}$$

and

$$|a_0(y,\eta,\lambda)| \le c(1+|\eta|^s) \tag{A3.3}$$

for $(y,\eta,\lambda) \in Y \times F \times \Lambda$. Then we consider the parameter-dependent semilinear elliptic equation

$$\mathcal{A}(\lambda)u + \mathcal{B}(u,\lambda)u = f(u,\lambda)\ , \tag{A3.4}$$

where

$$\mathcal{A}(\lambda) := -\sum_{j,k=1}^{n} a_{j,k}(\cdot,\lambda)\partial_j\partial_k\ ,\quad \mathcal{B}(u,\lambda) := \sum_{j=1}^{n} a_j(\cdot,u,\lambda)\partial_j + a_0(\cdot,u,\lambda)\ .$$

By a solution of (A3.4) we mean a pair $(u,\lambda) \in C^2(\overline{Y},F) \times \Lambda$ satisfying (A3.4) point-wise. (Of course, weaker concepts of solutions are possible. But being interested in boundedness properties, we leave aside regularity questions.)

A3.1 Theorem. *Suppose that $\mathcal{S}$ is a set of solutions of* (A3.4) *and there exists $p_0 > 1$ with $p_0 \ge sn/2$ such that one of the following conditions is satisfied*:

(i) $p_0 > sn/2$ *and $\mathcal{S}$ is bounded in $L_{p_0}(Y,F) \times \Lambda$;*

(ii) $p_0 = sn/2$ *with $n \ge 3$ and $\mathcal{S}$ is compact in $L_{p_0}(Y,F) \times \Lambda$.*

Then $\mathcal{S}$ is bounded in $W^2_{p,\mathrm{loc}}(Y,F) \times \Lambda$ for each $p \in [1,\infty)$, hence in $C^\alpha(Y,F) \times \Lambda$ for each $\alpha \in [0,2)$.

Proof. Since Λ is bounded we have to show that, given any open $X \subset\subset Y$, the set $\mathcal{U} := \{\, u\ ;\ (u,\lambda) \in \mathcal{S} \,\}$ is bounded in $W^2_p(X,F)$ for each $p \in [1,\infty)$. Then the second assertion follows from Sobolev's embedding theorem.

From (A3.1), the compactness of $\overline{Y} \times \Lambda \times S^{n-1}$, and the upper semicontinuity of the spectrum we easily infer that $\mathcal{A}(\lambda)$ is uniformly regularly elliptic for $\lambda \in \Lambda$.

Set $p := p_0/s > n/2$ if (i) is satisfied, and fix $p > 1$ with $n/4 < p < n/2$ if (ii) is true. Then

$$\big(\lambda \mapsto \mathcal{A}(\lambda)\big) \in C\big(\Lambda, \mathcal{E}\ell\ell_p(Y)\big)\ . \tag{A3.5}$$

Put $q_1 := 2p_0/s$ and $q_0 := p_0/s$, so that $\mathbb{E}_{p,\vec{q}}(Y)$ is specified. It is a well-known consequence of (A3.2) and (A3.3) that the map

$$L_{p_0}(Y,F) \times \Lambda \mapsto \mathbb{E}_{p,\vec{q}}(Y)\ ,\quad (u,\lambda) \mapsto \mathcal{B}(u,\lambda) \tag{A3.6}$$

is continuous and bounded on bounded sets.

Now put $\mathfrak{A} := \{\, \mathcal{A}(\lambda)\ ;\ \lambda \in \Lambda \,\}$ if (i) is satisfied, and

$$\mathfrak{A} := \{\, \mathcal{A}(\lambda) + \mathcal{B}(u,\lambda)\ ;\ (u,\lambda) \in \mathcal{S} \,\}$$

if (ii) is true. Also set $\mathfrak{B} := \{\, \mathcal{B}(u,\lambda)\ ;\ (u,\lambda) \in \mathcal{S} \,\}$ if (i) is valid, and $\mathfrak{B} := \{0\}$ otherwise. Then it follows from (A3.5), (A3.6), and our assumptions on $\mathcal{S}$ that $\mathfrak{A}$ is compact in $\mathcal{E}\ell\ell_p(Y)$ and $\mathfrak{B}$ is bounded in $\mathbb{E}_{p,\vec{q}}(Y)$.

Let $X \subset\subset Y$ be given and choose open sets X_j with

$$Y \supset\supset X_1 \supset\supset X_2 \supset\supset X_3 \supset\supset X\ .$$

Then Theorem A2.1 implies that $\mathcal{U}$ is bounded in $W_p^2(X_1, F)$, thanks to the boundedness of $\{ f(\cdot, \lambda) \ ; \ \lambda \in \Lambda \}$ in $L_p(Y, F)$ and the boundedness of $\mathcal{U}$ in $L_{p_0}(Y, F)$, hence in $L_p(Y, F)$. Then we infer from (A1.5) that $\mathcal{U}$ is bounded in $L_r(X_2, F)$, where $r := \infty$ if (i) is satisfied, and $1/r := 1/p - 2/n < 2/n$ if (ii) is true. In the latter case we can choose p arbitrarily close to $n/2$ so that $p_1 := r/s > n/2$. Then, by replacing Y by X_2 and p_0 by p_1, respectively, we are again in the situation of case (i). Repeating the above argument we find that $\mathcal{U}$ is bounded in $L_\infty(X_3, F)$ in case (i) as well as in case (ii).

Now we fix any $p \in (n, \infty)$ and specify $\mathbb{E}_{p,\vec{q}}(X_3)$ by setting $q_1 := q_0 := p$. Then $\mathfrak{A} := \{ \mathcal{A}(\lambda) \ ; \ \lambda \in \Lambda \}$ is a compact subset of $\mathcal{E}\ell\ell_p(X_3)$ and

$$\mathfrak{B} := \{ \mathcal{B}(u, \lambda) \ ; \ (u, \lambda) \in \mathcal{S} \}$$

is a bounded subset of $\mathbb{E}_{p,\vec{q}}(X_3)$. Hence a further application of Theorem A2.1 guarantees that $\mathcal{U}$ is bounded in $W_p^2(X, F)$, which proves the theorem. □

A3.2 Remark. If Y has a C^2-boundary and we specify boundary conditions such that we obtain a regularly elliptic boundary value problem, an analogue to Theorem A3.1 for estimates up to the boundary is valid. For this we refer to [A3]. □

Acknowledgments. The authors thank DGICYT and DGES of Spain for research support under grants DGICYT PB93-0465 and PB96-0621, respectively, and the Swiss National Science Foundation as well.

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