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Abstract methods in differential equations

H. Amann

Abstract. This is an expanded version, enriched by references, of my inaugural speech held on November 7, 2001 at the Real Academia de Ciencas Exactas, Físicas y Naturales in Madrid. It explains in a nontechnical way, accessible to a general scientific community, some of the motivation and basic ideas of my research of the last twenty years on a functional-analytical approach to nonlinear parabolic problems.

Métodos abstractos en ecuaciones diferenciales

Resumen. El presente trabajo corresponde a una versíon ampliada de la exposicíon del autor tenida el 7 de Noviembre 2001 en la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid con motivo de su nombramiento como miembro extranjero de esta institucíon. Evitando detalles técnicos, este artículo panorámico expone algunas de las motivaciones e ideas fundamentales de las investigaciones del autor durante los últimos veinte años sobre el elfoque analítico-funcional de problemas parabólicos no lineales.

1. Introduction

Since my days as a student I have been attracted by the fact that mathematical models can help to reveal hidden connections and lead to rational explanations for complex, and sometimes unexpected, natural, behavioral or other phenomena which, at a first glance, seem to be mysterious and incomprehensible.

Most mathematical models involve differential equations, predominantly systems of partial differential equations, of an almost prohibitive complexity. For the investigation of systems of this type it is most important to have a guiding line which leads through the jungle of technicalities to the heart of the matter. Very often such a guiding line can be obtained by embedding the given problems in a larger and more abstract class in which the technical difficulties are hidden so that one has a better chance to discover the general underlying structure and properties. By this way one is often led to develop tools which not only help to solve the given problem, but apply equally well to many others, being seemingly unrelated.

Of course, this is not only true for the field of differential equations, but also for almost all parts of mathematics. The benefits of such an approach have been clearly expressed by one of the great masters of abstraction, David Hilbert. In his lecture at the International Congress of Mathematicians in Paris in the year 1900 [25] one reads:

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> Aber — so fragen wir — wird es bei der Ausdehnung des mathematischen Wissens für den einzelnen Forscher nicht schließlich unmöglich, alle Teile dieses Wissens zu umfassen? Ich möchte als Antwort darauf hinweisen, wie sehr es im Wesen der mathematischen Wissenschaft liegt, daß jeder wirkliche Fortschritt stets Hand in Hand geht mit der Auffindung schärferer Hilfsmittel und einfacherer Methoden, die zugleich das Verständnis früherer Theorien erleichtern und umständliche ältere Entwicklungen beseitigen, und daß es daher dem einzelnen Forscher, indem er sich diese schärferen Hilfsmittel und einfacheren Methoden zu eigen macht, leichter gelingt, sich in den verschiedenen Wissenszweigen der Mathematik zu orientieren, als dies für irgend eine andere Wissenschaft der Fall ist.[1]

In the following, I shall try to give an idea of some of my research interests of the last twenty years. During that period I was predominantly concerned with a functional-analytical approach to parabolic evolution equations. In my opinion, functional analysis, combined with so-called "hard analysis" and many other mathematical subjects, is particularly well suited for providing an abstract, powerful, and sufficiently general framework for the study of nonlinear partial differential equations.

Of course, it is well-known that there are intimate connections between functional analysis and partial differential equations. In fact, large parts of linear functional analysis have been developed in order to provide the abstract tools for an efficient and unified study of linear partial differential equations. The point I want to make is that functional analysis is also very useful for the investigation of nonlinear differential equations.

Clearly, I shall only be able to scratch the surface, and most of the time I shall be rather vague. In particular, I shall not address specialists. Nevertheless, I hope that I can give an idea of "what is behind the scene".

2. A model case: reaction-diffusion systems

I begin with a model problem to illustrate the origin of some of the equations and to point out the principal questions which have to be addressed.

Let Ω be a bounded domain in $\mathbb{R}^n$, where $n = 1, 2$ or 3 in most physical relevant situations, with a smooth boundary Γ and outer unit normal ν. Suppose that Ω is occupied by some species (say a chemical substance, a population, heat, etc.) with which we can associate a space and time dependent density u. Also suppose that it can move within Ω and that its movement is determined by a flux vector(field) $\vec{\jmath}$. Assume that the distribution of the species throughout Ω at a given time, say at $t = 0$, is described by the "initial density" u^0 and that one wants to predict this distribution at any given later time t.

The basic hypothesis which we impose is the law of conservation of mass saying that

> at any given time the increase of total mass contained in a subdomain B of Ω equals the mass flowing through the boundary ∂B into B, augmented by the mass being produced within B.

Thus, denoting by f the "production density" (which may change its sign throughout Ω) and by $\vec{n}$ the outer unit normal of ∂B, this conservation law takes the form

$$\partial_t \int_B u\,dx = -\int_{\partial B} \vec{\jmath} \cdot \vec{n}\,d\sigma + \int_B f\,dx, \tag{1}$$

where $d\sigma$ is the volume measure of ∂B. By converting the surface integral into a volume integral by means of Gauss' theorem and by letting B shrink to a given point in Ω, we deduce from (1) that the density u

[1] But — so we ask — given the expansion of the mathematical knowledge, will it eventually not be impossible for the individual researcher to encompass all parts of this knowledge? As an answer I want to point out that it is a basic feature of mathematics that every real progress is intimately tied to the discovery of more efficient tools and simpler methods which also facilitate the comprehension of earlier theories and put away with cumbersome older developments, and that, consequently, the individual researcher will be able, by acquiring these more efficient tools and simpler methods, to find his way through the different branches of mathematics more easily than this is the case for any other scientific field.

satisfies the infinitesimal conservation law

$$\partial_t u + \nabla \cdot \vec{\jmath} = f \quad \text{in } \Omega\,, \qquad t > 0, \tag{2}$$

where $\nabla\cdot$ denotes "divergence".

On the boundary of Ω we assume that the inflow is prescribed, that is,

$$-\nu \cdot \vec{\jmath} = g \quad \text{on } \Gamma\,, \qquad t > 0. \tag{3}$$

In general, the flux vector $\vec{\jmath}$ is related to the density u by means of a constitutive law. A frequent and intuitive assumption, being justified in many cases by basic physical considerations, says that mass moves from places of higher density to those of a lower one along the lines of steepest descent. In other words,

$$\vec{\jmath} = -a\nabla u, \tag{4}$$

where ∇u is the gradient of u, and a, the diffusion coefficient, is a positive function. In concrete applications, (4) is known as Fick's law, Fourier's law, etc., according to the given framework.

Inserting (4) into (2) and (3) leads to the following initial boundary value problem for u:

$$\begin{aligned} \partial_t u - \nabla\cdot(a\nabla u) &= f \quad && \text{in } \Omega\times(0,\infty),\\ a\,\partial_\nu u &= g && \text{on } \Gamma\times(0,\infty),\\ u(\cdot,0) &= u^0 && \text{on } \Omega. \end{aligned} \tag{5}$$

In the particular case where $a = 1$, system (5) reduces to the initial value problem for the heat equation with Neumann boundary conditions:

$$\begin{aligned} \partial_t u - \Delta u &= f \quad && \text{in } \Omega\times(0,\infty),\\ \partial_\nu u &= g && \text{on } \Gamma\times(0,\infty),\\ u(\cdot,0) &= u^0 && \text{on } \Omega. \end{aligned} \tag{6}$$

In the general case the differential equation in (5) is of parabolic type.

Of course, in a more realistic model the diffusion coefficient as well as the "exterior densities" depend on u as well. Thus we are led to investigate the quasilinear parabolic initial boundary value problem

$$\begin{aligned} \partial_t u - \nabla\cdot\big(a(u)\nabla u\big) &= f(u) \quad && \text{in } \Omega\times(0,\infty),\\ a(u)\partial_\nu u &= g(u) && \text{on } \Gamma\times(0,\infty),\\ u(\cdot,0) &= u^0 && \text{on } \Omega. \end{aligned} \tag{7}$$

Whereas the linear problem (5) is well-studied, we are far from a complete understanding of the quasilinear problem (7), although much research is presently devoted to the investigation of problems of this type.

In most mathematical models of relevant physical (chemical, biological, sociological, etc.) situations one has to consider systems involving several (say N) species moving within Ω and interacting with each other. The basic hypotheses are in this case the assumptions that with each species k we can be associate a density u_k and that the law of conservation of mass holds for each species k. Thus we arrive at the system of N conservation laws

$$\begin{aligned} \partial_t u_k + \nabla\cdot\vec{\jmath}_k &= f_k \quad && \text{in } \Omega\times(0,\infty),\\ -\nu\cdot\vec{\jmath}_k &= g_k && \text{on } \Gamma\times(0,\infty) \end{aligned} \tag{8}$$

for $1 \le k \le N$. In general, the flux vector $\vec{\jmath}_k$ as well as the exterior densities f_k and g_k of species k may depend on u_k and on all other species as well, that is, on the full vector $u := (u_1,\dots,u_N)$. In this case a natural generalization of the constitutive law (4) says that the flux vector $\vec{\jmath}_k$ depends linearly on the gradients ∇u_k for $1 \le k \le N$, that is,

$$\vec{\jmath}_k(u) = -a_{k1}\nabla u_1 - \cdots - a_{kN}\nabla u_N\,, \qquad 1 \le k \le N,$$

where, in general, the diffusion coefficients a_{ki} depend nonlinearly on u. In a great many applications this hypothesis can be justified on the basis of thermodynamical considerations (cf. [21]). Thus in this case — and restricting ourselves to the case $N = 2$, for simplicity — we obtain the following system of reaction-diffusion equations:

$$\left.\begin{aligned}\partial_t u_1 - \nabla\cdot\bigl(a_{11}(u)\nabla u_1 + a_{12}(u)\nabla u_2\bigr) &= f_1(u)\\ \partial_t u_2 - \nabla\cdot\bigl(a_{21}(u)\nabla u_1 + a_{22}(u)\nabla u_2\bigr) &= f_2(u)\end{aligned}\right\}\quad \text{in } \Omega\times(0,\infty), \tag{9}$$

subject to the boundary conditions

$$\left.\begin{aligned}a_{11}(u)\partial_\nu u_1 + a_{12}(u)\partial_\nu u_2 &= g_1(u)\\ a_{21}(u)\partial_\nu u_1 + a_{22}(u)\partial_\nu u_2 &= g_2(u)\end{aligned}\right.\quad \text{on } \Gamma\times(0,\infty), \tag{10}$$

which has to be complemented by the known initial condition

$$u(\cdot,0) = u^0 := (u_1^0, u_2^0) \qquad \text{on } \Omega. \tag{11}$$

By using vector notation, setting $f := (f_1, f_2)$ and $g := (g_1, g_2)$, and introducing the matrix of diffusion coefficients

$$a(u) := \begin{bmatrix} a_{11}(u) & a_{12}(u)\\ a_{21}(u) & a_{22}(u)\end{bmatrix},$$

system (9)–(11) can be concisely represented in the form

$$\begin{aligned}\partial_t u - \nabla\cdot\bigl(a(u)\nabla u\bigr) &= f(u) &&\text{in } \Omega\times(0,\infty),\\ a(u)\partial_\nu u &= g(u) &&\text{on } \Gamma\times(0,\infty),\\ u(\cdot,0) &= u^0 &&\text{on } \Omega.\end{aligned} \tag{12}$$

Superficially, this system is equal to (7). But now we have to keep in mind that u, f, and g are N-vectors, and a is an $(N\times N)$-matrix (and $\nabla\cdot\bigl(a(u)\nabla u\bigr)$ has to be interpreted in the obvious sense suggested by the generalization of (9) to the case of N densities).

Whereas in the scalar case the assumption that the diffusion coefficient be positive, is both natural and simple, a priori it is not clear what could be the correct generalization of this concept in the $(N\times N)$-matrix case. Formally, one might be tempted to assume here also that $a > 0$, where this inequality is now to be interpreted to mean that a is symmetric and positive definite. However, this hypothesis is much too restrictive for many applications to concrete problems of science.

As an example we consider a (2×2)-predator-prey system. In other words, we consider a population model in which u_1 is the density of a prey and u_2 that of a predator. More precisely, we assume that the flux vectors are of the particular form

$$\begin{aligned}\vec{\jmath}_1(u) &= -\alpha(u)\nabla u_1 - \beta(u)\nabla u_2,\\ \vec{\jmath}_2(u) &= \quad\beta(u)\nabla u_1,\end{aligned} \tag{13}$$

where $\alpha(u) > 0$ and $\beta(u) > 0$. This allows for the following heuristic interpretation: thanks to the term $-\alpha(u)\nabla u_1$, the prey moves from places of high density of his own species to those of lower ones, that is, the prey "wants to stay away from places of concentration of his own kind". The term $-\beta(u)\nabla u_2$ means that, in addition, the prey "tries to stay away from places of high concentration of predators". On the other hand, the form of the flux vector $\vec{\jmath}_2$ of the predator implies that the latter moves towards places of high concentration of prey. This is certainly a rather realistic behavior in a predator-prey system, and it is desirable that our theory be general enough to embrace such cases. Clearly, the matrix

$$a := \begin{bmatrix} \alpha & \beta\\ -\beta & 0\end{bmatrix}, \qquad \alpha,\beta > 0,$$

does not satisfy $a > 0$ in the sense of positive definiteness.

In the case (5) of a scalar unknown, in particular in the case of the heat equation (6), it is known that the corresponding parabolic equation possesses an important smoothing property guaranteeing that any solution is infinitely smooth in Ω for $t > 0$ if f has this property, no matter how irregular its initial value u^0 may be. This is one of the basic features of parabolic equations which we want to preserve in the case of $(N \times N)$-systems. Thus we have to find hypotheses on a guaranteeing this property and being general enough to embrace the simple predator-prey system (13). It turns out that the correct generalization to $(N \times N)$-systems is a general concept of parabolicity to be discussed in more detail below.

Thus the *reaction-diffusion system* (12) is a particular instant of a general *parabolic initial boundary value problem.* It exhibits two features adding substantially to its complexity:

- the system is quasilinear, meaning that
  the diffusion coefficients depend nonlinearly on the solution itself;
- the boundary conditions are nonlinear as well.

3. Natural questions

Clearly, given a quasilinear parabolic initial boundary value problem, there arise a number of natural question. In the first place:

(A) *Is problem* (12) *well-posed in the sense of Hadamard, that is, does it posses a unique solution depending continuously on the data?*

Since we are interested in problems which are mathematical models for complex phenomena of science (engineering, sociology, etc.), well-posedness is essential. Should it turn out that a given system is not well-posed, it cannot be a valid model for describing and predicting realistic phenomena.

Having established well-posedness of system (12), there occur immediately further questions concerning the qualitative behavior of its solutions. First of all:

(B) *Do some (or all) solutions exist globally, that is, for all times, or does there occur "blow-up" in finite time? In other words: is the model capable of describing long-time phenomena?*

Ideally, one would like to get a complete description of the "flow" generated by (12) on an appropriate "phase space", that is, the space where the solution $u(\cdot, t)$ lives during its existence. It is well-known that this is not even possible in the much simpler case of ordinary differential equations, in general. Thus one has to be more modest and ask for particularly simple structures in the "phase portrait". For example:

(C) *Do there exist critical points, periodic orbits, limit cycles, etc.?*

Note that even these questions are enormously difficult since in our problem the phase space is an infinite-dimensional space of functions, in general. For example, the question of the existence of a critical point of (12) amounts to the problem of guaranteeing the solvability of the nonlinear (elliptic) system

$$\begin{aligned} -\nabla \cdot \big(a(u)\nabla u\big) &= f(u) \quad \text{in } \Omega, \\ a(u)\partial_\nu u &= g(u) \quad \text{on } \Gamma. \end{aligned}$$

Having found simple orbits, like critical points or periodic orbits, there arise naturally questions of stability:

(D) *Is a given critical point or periodic orbit stable? If not: can one characterize its amount of instability, say dimensions of unstable manifolds?*

A much more ambitious question is the one concerning the stability of the phase portrait under variations of parameters in the equations, that is, of the functions f, g, and a:

(E) *Is system* (12) *structurally stable?*

The questions raised in (A)–(E) are of fundamental theoretical and practical importance. It is obvious that they are exceedingly difficult and form a program of research for generations to come. It is also clear that even partial answers to those questions will be of immense value since they can contribute to a deeper understanding not only of mathematics but also of nature and the world we are living in.

A well-developed theory of quasilinear parabolic systems is also the basis for the use of these models to manipulate a given setting to achieve a more desirable situation. To illustrate this we return to the predator-prey system defined by (13). The heuristic considerations given above indicate that that system can describe situations in which "a predator chases a prey which runs away from it". In such a case one could be interested in a stable situation in the sense that a periodic behavior prevails. In other words: one can ask if exterior data f_k and g_k can be chosen in such a way that problem (8), that is, problem (9), (10) with the flux vectors $\vec{\jmath}_1$ and $\vec{\jmath}_2$ given by (13), possesses periodic solutions. Since f and g depend on u, in general, this amounts to a *feed-back control problem* for (12).

4. Weak solutions and evolution equations

Unlike in the case of ordinary differential equations, in the theory of partial evolution equations there is no natural state space in which (12) is to be considered. The correct choice of the state space is of predominant importance and crucial for the success of the investigation.

Unfortunately, spaces of continuously differentiable functions — seemingly the most natural candidates — are "not good" in a very precise sense, as is well-known in the theory of partial differential equations. This is related to the fact that the fundamental Laplace operator Δ does not define an isomorphism from

$$C_\gamma^2(\overline{\Omega}) := \left\{ u \in C^2(\overline{\Omega}) \; ; \; u \,|\, \Gamma = 0 \right\}$$

onto $C(\overline{\Omega})$, although it maps $C_\gamma^2(\overline{\Omega})$ injectively into $C(\overline{\Omega})$ and has a dense image. For the same reason spaces of distributions whose derivatives of low order are integrable, that is, Sobolev spaces $W_1^k(\Omega)$, are "not good".

"Good" spaces, as far as isomorphism theorems for linear elliptic problems — or more fundamentally: Fourier multiplier theorems — are concerned, are L_p-Sobolev spaces, that is, spaces $W_p^k(\Omega)$, for $1 < p < \infty$.

Whereas in the theory of linear partial differential equations there is no real need to resort to the more difficult cases for which $p \neq 2$, for nonlinear equations the possibility to work with $p \neq 2$ is crucial. The correct choice of p is dictated by the concrete problem and depends on the growth of the nonlinear functions f and g at infinity. If p is inappropriately selected, say if one chooses $p = 2$ in order to use the advantages of an easy Hilbert space setting, (12) will, in general, not be well-posed.

Having proven the well-posedness of (12) in an appropriate L_p-setting, the study of the long-time behavior of its solutions, that is, of the flow generated by it, requires, as a rule, the establishing of a priori bounds for some or all solutions. It is clear that it is much easier to find such bounds in spaces of low regularity than in spaces of regular functions. For example, in concrete problems, physical laws like the conservation of mass, energy, etc. sometimes guarantee that integral averages of the solution, say $\int_\Omega u(x,t)\,dx$ or $\int_\Omega u^2(x,t)\,dx$, remain bounded throughout the evolution, whereas it may be difficult or even impossible to establish the boundedness of integral norms (let alone uniform norms) involving spacial derivatives. This suggests to construct a solvability theory for (12) and similar problems in a generalized sense, that is, in the sense of distributions, as it is known from the theory of linear partial differential equations (e.g., [27]).

However, locally convex spaces of distributions, in particular the space $\mathcal{D}(\Omega)$ of Schwartz distributions, are "not good" for nonlinear problems. First: this is due to the fact that it is not possible to define a "point-wise" product of two distributions, in general. Second: locally convex spaces which are not Banach spaces are not suitable for nonlinear analysis, mainly because the inverse function theorem fails to hold in such spaces.

Thus one has to find a setting which, on the one hand, involves spaces of low regularity and, on the other hand, uses Banach spaces of distributions which are not too singular. Of course, this necessitates a concept of solvability which does not involve classical but distributional derivatives. Consequently, one is led to look for *weak solutions* rather than for classical ones.

Originating in the work of Leray [30], Friedrichs [24], Sobolev [35], and others, a concept of weak solutions is well understood and easy to explain:

Suppose that u is a classical solution of (12). Then, given a "test function" $\varphi \in C^1(\overline{\Omega})$, we multiply the first equation in (12) by φ, integrate over Ω, integrate by parts by means of Green's formula, and, using obvious notation, arrive at

$$\int_\Omega \varphi\, \partial_t u\, dx + \int_\Omega \nabla\varphi a(u)\nabla u\, dx = \int_\Omega \varphi f(u)\, dx + \int_\Gamma \varphi g(u)\, d\sigma.$$

Setting

$$\langle v, w\rangle := \int_\Omega vw\, dx\ , \quad \langle v, w\rangle_\Gamma := \int_\Gamma vw\, d\sigma,$$

this relation can be expressed more succinctly by

$$\langle \varphi, \partial_t u\rangle + \big\langle \nabla\varphi, a(u)\nabla u\big\rangle = \big\langle \varphi, f(u)\big\rangle + \big\langle \varphi\,|\,\Gamma, g(u)\big\rangle_\Gamma, \tag{14}$$

which has to hold for each $\varphi \in C^1(\overline{\Omega})$. Now a function u is said to be a *weak solution* of (12), provided u is such that each term of (14) is well-defined for $\varphi \in C^1(\overline{\Omega})$ and such that (14) holds for every $\varphi \in C^1(\overline{\Omega})$. Thus every classical solution is a weak one, whereas the converse does not hold, in general.

Observe that the concept of weak solutions is much closer to the physical relevant formulation (1) than this is the case for the (mathematically idealized) infinitesimal conservation law (2).

In order to understand the concept of weak solutions better, first assume that w is a given function in $L_\infty(\Omega)$ (where, here and below, we do not indicate notationally that our function spaces are either spaces of $\mathbb{R}^N$-valued or $\mathbb{R}^{N\times N}$-valued functions, depending on the context). Then $a(w)$ belongs to $L_\infty(\Omega)$ also, so that $\big\langle \nabla\varphi, a(w)\nabla v\big\rangle$ is well-defined if $\nabla\varphi$ and ∇v are square integrable. This suggests to introduce the Sobolev space $V := H^1(\Omega)$ of all functions in $L_2(\Omega)$ whose first derivatives belong to $L_2(\Omega)$ as well. Then

$$\big((\varphi, v) \mapsto \big(\nabla\varphi, a(w)\nabla v\big)\big) : V \times V \to \mathbb{R} \tag{15}$$

is a continuous bilinear map. Observe that $V \overset{d}{\hookrightarrow} H := L_2(\Omega)$, where $\overset{d}{\hookrightarrow}$ means continuous and dense embedding. Consequently, $H \overset{d}{\hookrightarrow} V'$, where V' is the dual space of V, and we see that $V \overset{d}{\hookrightarrow} V'$. Elementary functional analysis guarantees the existence of a unique bounded linear operator $\mathcal{A}(w)$ from V into V', that is,

$$\mathcal{A}(w) \in \mathcal{L}(V, V'),$$

such that

$$\big\langle \varphi, \mathcal{A}(w)v\big\rangle_{V'} = \big\langle \nabla\varphi, a(w)\nabla v\big\rangle\ , \qquad \varphi, v \in V, \tag{16}$$

where $\langle\cdot,\cdot\rangle_{V'}$ denotes the duality pairing between V' and $(V')' = V$. Similarly, by using the trace theorem from the theory of Sobolev spaces and imposing suitable restrictions on f and g, one finds that there exists a unique $\mathcal{F}(w) \in V'$ satisfying

$$\big\langle \varphi, \mathcal{F}(w)\big\rangle_{V'} = \big\langle \varphi, f(w)\big\rangle + \big\langle \varphi\,|\,\Gamma, g(w)\big\rangle_\Gamma\ , \qquad \varphi \in V.$$

We do not go into detail concerning hypotheses for f and g since, in our models, the operator $\mathcal{F}(w)$ is (in a certain precise sense) less important than the bilinear term (16) involving spacial derivatives. Hence we concentrate on that latter term.

Lastly, interpreting $\partial_t u$ as an element, $\dot{u}$, of V', one sees that (14) takes the form

$$\langle \varphi, \dot{u}\rangle_{V'} + \big\langle \varphi, \mathcal{A}(u)u\big\rangle_{V'} = \big\langle \varphi, \mathcal{F}(u)\big\rangle_{V'}\ , \qquad \varphi \in V,$$

which can be expressed equivalently by saying that u is a solution of the initial value problem

$$\dot{u} + \mathcal{A}(u)u = \mathcal{F}(u) , \quad t > 0 , \qquad u(0) = u^0 \tag{17}$$

in the Hilbert space V'. Thus we have found a formulation of our model problem (12) as an ordinary differential equation — a nonlinear evolution equation — in a Banach space, namely in V'.

5. Why classical methods fail

Now we have to find conditions guaranteeing the solvability of (17). A standard hypothesis in the theory of continuous bilinear forms in Hilbert spaces is a coercivity requirement. It amounts to assuming that there exists a constant $\alpha > 0$ such that

$$\big\langle v, \mathcal{A}(w)v \big\rangle_{V'} \geq \alpha \, \|v\|_V^2 , \qquad v \in V , \quad w \in L_\infty(\Omega). \tag{18}$$

It is not difficult to verify that this uniform coercivity condition implies, in the case of our model problem (12), that the matrix of diffusion coefficients satisfies the strong ellipticity condition

$$a(w) + a(w)^\top \geq 2\alpha , \qquad w \in C(\overline{\Omega}). \tag{19}$$

Now suppose that condition (18) is satisfied. Then, in order to solve problem (17), it is natural to consider two steps.

In *step* (i) we linearize the equation in (17) by fixing $T > 0$ and

$$w \in L_\infty\big((0,T), L_\infty(\Omega)\big) \tag{20}$$

and inserting w in all nonlinear terms. Thus we arrive at the linear initial value problem

$$\dot{v} + \mathcal{A}(w)v = \mathcal{F}(w) , \quad 0 < t \leq T , \qquad v(0) = u^0. \tag{21}$$

Thanks to hypothesis (18) we can employ a Galerkin method, as introduced in the theory of evolution equations by E. Hopf [26] in his famous paper on the global weak solvability of the Navier-Stokes equations from mathematical fluid mechanics, and as widely exploited and popularized by J.L. Lions [31], O.A. Ladyzhenskaya [29], and others. This method provides us with a unique solution

$$v := v(w) \in L_2\big((0,T), V\big) \cap H^1\big((0,T), V'\big)$$

of (21).

Step (ii) then consists in showing that the map

$$w \mapsto v(w) \tag{22}$$

possesses a fixed point, which is obviously a solution of the nonlinear problem (12).

However, this approach poses serious difficulties, namely:

• *problem* (1): The solution of the linearized equation will not belong to the same class as w, in general, that is,

$$v(w) \notin L_\infty\big((0,T), L_\infty(\Omega)\big),$$

in general. Consequently, the fixed point map (22) does not map back into its domain. Thus it is not possible to apply methods from topology or nonlinear functional analysis guaranteeing the existence of a fixed point. In other words: step (ii) does not work, in general.

The situation is much better if $\mathcal{A}$, that is a, is independent of u since in that case the regularity of (20) is not needed. Thus it is possible to use the above scheme to prove the well-posedness of the *semilinear* initial value problem

$$\dot{u} + \mathcal{A}u = \mathcal{F}(u) , \quad t > 0 , \qquad u(0) = u^0,$$

which is induced by the semilinear parabolic system obtained from (12) in the case where a is independent of u. In fact, most of the papers published on semilinear parabolic problems employ the popular Galerkin approach of step (i).

But, even restricting us to the narrow, but still important subclass of semilinear problems, we now arrive at

• *problem* (2): Keeping in mind that we are predominantly interested in *systems*, it turns out that the coerciveness assumption (18) is too restrictive and not satisfied in most important problems originating in physics. For example, condition (19) is obviously not valid for the simple predator-prey system (13).

There are other methods for handling nonlinear evolution equations, namely the theory of monotone operators going back to Minty and Browder, and the theory of accretive operators which has been developed by Bénilan, Crandall, and others (see [37] for detailed explanations of those methods, as well as for concrete examples). Although both theories apply to many problems of interest and then yield excellent results, they do not cover systems of reaction-diffusion equations and related problems, and require coerciveness. Except for very particular circumstances, *systems of parabolic equations do induce neither monotone nor accretive operators. In particular, they are not coercive.* Thus a different approach is required.

6. L_p-theory

In analyzing the failure of the linearization approach outlined in the previous section we see that one of the reasons why it does not work lies in the fact that the solution $v(w)$ of the linearized problem (21) does not possess enough regularity to yield a well-defined fixed point problem (22). However, better regularity can easily be obtained.

For this we return to the definition of the linear operator $\mathcal{A}(w)$ via the bilinear form (15). We again fix $w \in C(\overline{\Omega})$ and also fix a real number $p > n$. Then it is easily verfied that

$$\big((\varphi, v) \mapsto \langle \nabla\varphi, a(w)\nabla v\rangle\big) : \ H^1_{p'} \times H^1_p \to \mathbb{R} \tag{23}$$

is a well-defined continuous bilinear form, where $H^1_q := H^1_q(\Omega) = W^1_q(\Omega)$ is the usual Sobolev space of order 1 for $1 < q < \infty$, and $1/q + 1/q' = 1$. Now we note that

$$X_1 := H^1_p \overset{d}{\hookrightarrow} L_p\ , \quad H^1_{p'} \overset{d}{\hookrightarrow} L_{p'}\,.$$

Thus, setting $X_0 := (H^1_{p'})'$ with respect to the duality pairing naturally induced by $\langle\cdot,\cdot\rangle$, that is, the L_p-duality pairing, it follows that

$$X_1 \overset{d}{\hookrightarrow} L_p \overset{d}{\hookrightarrow} X_0.$$

Consequently, (23) implies the existence of a unique

$$A(w) \in \mathcal{L}(X_1, X_0)$$

satisfying

$$\big\langle \varphi, A(w)v\big\rangle_{X_0} = \big\langle \nabla\varphi, a(w)\nabla v\big\rangle\ , \qquad v \in X_1\ , \quad \varphi \in H^1_{p'}\,.$$

Similarly as in the L_2-case, given suitable assumptions for f and g, we can define a nonlinear map F by

$$\big\langle v, F(w)\big\rangle := \big\langle \varphi, f(w)\big\rangle + \big\langle \varphi|\Gamma, g(w)\big\rangle_\Gamma\ , \qquad \varphi \in H^1_{p'}. \tag{24}$$

Since $C^1(\overline{\Omega})$ is dense in $H^1_{p'}$, it follows that the weak form of (12) gives rise to an initial value problem

$$\dot{u} + A(u)u = F(u)\ , \quad t > 0\ , \qquad u(0) = u^0 \tag{25}$$

in the Banach space X_0. Thus (12) is now given a *weak L_p-formulation.* This harmless looking slight twist, the passage from L_2 to L_p, increases considerably the level of sophistication. But, as a reward, it yields a powerful method to deal with noncoercive problems.

Now we proceed similarly as above. We fix $T > 0$ and

$$w \in C\big([0,T], C(\overline{\Omega})\big) \tag{26}$$

and consider the linearized problem

$$\dot{v} + A(w)v = F(w)\ , \quad 0 < t \le T\ , \qquad v(0) = u^0. \tag{27}$$

Suppose that (27) has a unique solution

$$v(w) \in C\big([0,T], X_1\big) \cap C^1\big([0,T], X_0\big).$$

Then, since $p > n$ implies $X_1 \hookrightarrow C(\overline{\Omega})$ by Sobolev's embedding theorem, it follows that

$$v(w) \in C\big([0,T], C(\overline{\Omega})\big).$$

Thus (26) shows that the fixed point map $w \mapsto v(w)$ is well-defined in the Banach space $C\big([0,T], C(\overline{\Omega})\big)$. Hence step (ii) of the procedure outlined in Section 5 is now possible, in principle.

But now we are faced with the problem of solving the linear initial value problem (27). For this we first study an even simpler version where A is a constant map (note that $A(w)$ is an operator-valued function of t). In other words, suppose that

$$A \in \mathcal{L}(X_1, X_0)$$

and consider the linear problem

$$\dot{v} + Av = f\ , \quad t > 0\ , \qquad v(0) = u^0 \tag{28}$$

in X_0, where $f : \mathbb{R}^+ \to X_0$ is given (satisfying suitable restrictions).

In our present setting the Galerkin approach cannot be used anymore. In fact, since $p \neq 2$ (if $n \geq 2$), a coercivity assumption, which is the basis of that method, does not even make sense.

It is well-known that linear differential equations in general Banach spaces can be efficiently handled by the theory of operator semigroups. Recall that we are predominantly interested in reaction-diffusion systems containing as a particularly simple and important submodel the linear heat equation. One knows that the heat operator gives rise to a so-called analytic semigroup. This class of semigroups possesses an important smoothing property, being the abstract counterpart of the smoothing property inherent in the heat equation. Thus in the general case of (28) we impose the condition that

$$-A \text{ generates an analytic semigroup on } X_0.$$

Linear operators possessing this property can be characterized by the validity of a resolvent estimate of the form

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X_0)} \le c/|\lambda|\ , \qquad \operatorname{Re}\lambda \ge \omega, \tag{29}$$

where c and ω are suitable positive constants (and equation (29) is to be interpreted in its complexified version). Thus we have to prove that A satisfies (29), where A is defined by

$$\langle \varphi, Av\rangle_{X_0} := \langle \nabla\varphi, a\nabla v\rangle\ , \qquad v \in X_1\ , \qquad \varphi \in H^1_{p'}, \tag{30}$$

a being an $(N \times N)$-matrix-valued continuous function on $\overline{\Omega}$.

Note that X_0, the dual of $H^1_{p'}$, is not a space of distributions on Ω and that the definition of A is implicit. Thus it is by no means clear how to verify the validity of the resolvent estimate (29).

However, observe that (30) is related to the weak L_p-formulation of the linear parabolic problem (5) (in the case where u is an N-vector-valued function). In other words, A is induced by the "boundary value problem"

$$u \mapsto (\mathcal{A}_0 u, \mathcal{B}_0 u) := \big(-\nabla \cdot (a\nabla u), a\,\partial_\nu u\big).$$

With this boundary value problem we can associate its L_p-realization A_0, defined by

$$\mathrm{dom}(A_0) := H^2_{p,\mathcal{B}}\ , \quad A_0 u := \mathcal{A}u,$$

where

$$H^2_{p,\mathcal{B}} := \left\{\, u \in H^2_p(\Omega)\ ;\ a\,\partial_\nu u = 0 \,\right\} \overset{d}{\hookrightarrow} L_p(\Omega).$$

The operator A_0 can be studied by techniques from the theory of linear partial differential equations, using a combination of Fourier analysis in L_p-spaces, functional analysis, and hard analysis. By this way it is possible to establish resolvent estimates of type (29) for A_0 on the Banach space $L_p(\Omega)$, provided a satisfies the following *normal ellipticity* condition:

given any $x \in \overline{\Omega}$, all eigenvalues of the $(N \times N)$-matrix $a(x)$ possess strictly positive real parts. (31)

In fact, this condition is necessary and sufficient for the validity of (29) (cf. [1, Theorems 2.4 and 4.4]).

One easily sees that (31) is satisfied if the coercivity assumption (19) is true, that is, if the operator $\mathcal{A}_0$ is strongly uniformly elliptic. But, if $N > 1$, (31) is much weaker than (19). For example, condition (31) is always satisfied in the predator-prey problem (13).

It should be mentioned that (31) is satisfied iff the linear operator $\partial_t + \mathcal{A}_0$ is *Petrowski parabolic.*

7. Interpolation-extrapolation scales

According to the preceding considerations we know that we can solve (in principle) problem (28) if A is replaced by A_0. However, there are many good (technical and other) reasons for studying (12) in spaces weaker than $L_p(\Omega)$, particularly in X_0. We recall two of the most fundamental ones:

- a weak setting facilitates the derivation of a priori bounds;
- in a weak setting it is easy to incorporate nonlinear boundary conditions,

namely, by incorporating them into the right-hand side of (25) as is indicated by (24).

To render the weak L_p-approach successful, we have to derive estimate (29) in the "weak space" X_0. Surprisingly, this can be done by completely abstract techniques which apply to other, seemingly different, problems also:

We start with a reflexive Banach space E_0 and a densely defined closed linear operator A_0 in E_0 having a nonempty resolvent set. Then we can construct a variety of scales of Banach spaces and related operators

$$\left[\,(E_\alpha, A_\alpha)\ ;\ \alpha \in \mathbb{R}, \right],$$

so-called *interpolation-extrapolation scales* possessing the following properties:

(a) $E_\alpha \overset{d}{\hookrightarrow} E_\beta$ for $\alpha > \beta$;

(b) $A_\alpha \in \mathcal{L}(E_{\alpha+1}, E_\alpha),\ \alpha \in \mathbb{R}$;

(c) $-A_\alpha$ generates an analytic semigroup on E_α iff $-A_0$ possesses the corresponding property on E_0;

(d) $A_\beta \supset A_\alpha$ for $\alpha > \beta$.

Except for isomorphisms, the spaces E_α and operators A_α are uniquely determined by (E_0, A_0), provided $\alpha \in \mathbb{Z}$. For $\alpha \in \mathbb{R}\backslash\mathbb{Z}$ the corresponding spaces and operators are determined by interpolation techniques and depend on the interpolation methods being used. (We refer to [6, Chapter V] for a detailed study of the interpolation-extrapolation theory and to [3] for concrete examples.)

In the particular case where

$$E_0 := L_p(\Omega)\ , \quad A_0 := L_p\text{-realization of } (\mathcal{A}_0, \mathcal{B}_0),$$

it is possible to construct an interpolation-extrapolation scale such that

$$E_{1/2} = X_1 \ , \quad E_{-1/2} = X_0 \ , \quad A_{-1/2} = A,$$

where A is defined by (30), that is, A is the operator corresponding to the weak L_p-formulation of the boundary value problem $(\mathcal{A}_0, \mathcal{B}_0)$. This is due to the fact that for $\alpha < 0$ the pair (E_α, A_α) can be characterized by a fundamental duality theorem.

Hence it follows from (29), the fact that this estimate holds iff $\mathcal{A}_0$ is normally elliptic, and from (c) that

$-A$ generates an analytic semigroup on X_0
iff a satisfies the normal ellipticity condition (31).

Of course, a good solvability theory for (28) is just a first (and easiest) step on the way to a solvability theory for (25). Since the linearized operator $A(w)$ in (27) depends on $t \in [0, T]$, the next step is such a theory for nonautonomous problems

$$\dot{v} + A(t)v = f(t) \ , \quad 0 < t \leq T \ , \qquad v(0) = v^0. \tag{32}$$

In this connection it is important that mild regularity hypotheses for A and f are imposed only. These questions are studied in detail in the monograph [6].

8. Quasilinear evolution equations

By the preceding considerations we have put our model problem (12) into a general abstract framework which we can study independently of the concrete application. More precisely, we are led to investigate quasilinear evolution equations of the form

$$\dot{u} + A(u)u = F(u) \ , \quad t > 0 \ , \qquad u(0) = u^0 \tag{33}$$

in an arbitrary Banach space E, given the basic hypothesis that $A(w)$ is, for each fixed time-independent w, the negative generator of an analytic semigroup. By "quasilinearity" of (33) we mean that the nonlinearity F is subordinate to A, and the dependence of the linear operator $A(w)$ on w is also weaker than the dependence of $u \mapsto A(w)u$ on u. These vague formulations can be precisely quantified by studying (33) in appropriate interpolation-extrapolation scales.

Having at our disposal a good theory for the nonautonomous parabolic evolution problem (32), arguments basically known from the theory of ordinary differential equations can be adapted to our more general situation to prove that (33) generates a local semiflow, which means, in particular, that (33) is well-posed and depends continuously on the initial datum u^0. Precise formulations and properties of the semiflow can be found in [3].

It should be remarked that our approach is different from the "maximal regularity method" developed by Da Prato, Grisvard, and Lunardi and presented in detail in [32]. Whereas those authors handle "fully nonlinear equations" of the form

$$\dot{u} = f(u) \quad \text{in } (0, T] \ , \qquad u(0) = u^0$$

with f being a suitable nonlinear map in some Banach space, our approach is particularly well suited for the important class of quasilinear problems where it produces results which are superior to those obtained by other methods.

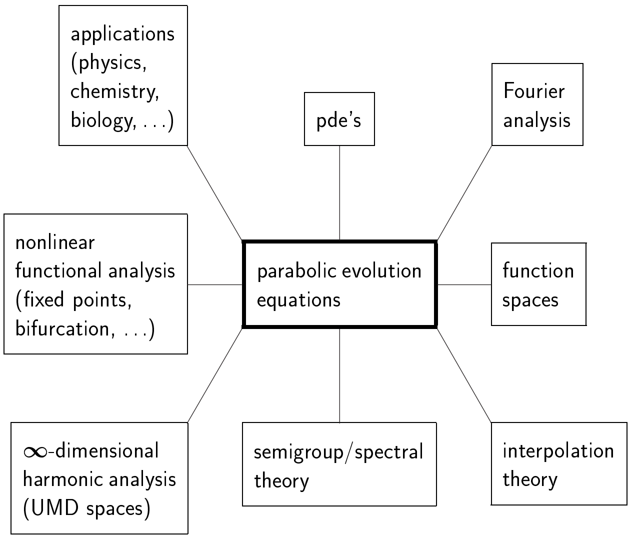
In the particular case where (33) is the abstract formulation of (12) it follows that the reaction-diffusion system (12) generates a local semiflow on the phase space $H^1_p(\Omega)$. This well-posedness result is the basis for qualitative studies addressing some of the questions outlined in (A)–(E) of Section 3. For example, by means of a Hopf type bifurcation analysis it can be shown that the feedback problem for the predator-prey system (13), referred to in Section 2 is solvable, that is, there exist f and g such that problem (8), with $\vec{\jmath}_k$ given by (13), possesses periodic solutions (cf. [2]).

It is clear that problem (33) embraces a multitude of concrete applications different from (12). Moreover, the interpolation-extrapolation technique is very flexible and gives rather precise results. Once a concrete problem has been given an abstract formulation of type (33), in order to establish its well-posedness it "only" remains to show that $-A(w)$ generates an analytic semigroup for each w in an appropriate phase space and the maps $w \mapsto A(w)$ and $w \mapsto F(w)$ are appropriately Lipschitz continuous. Of course, the "correct" choice of the interpolation-extrapolation space in which the abstract formulation is set up is of fundamental importance for the success of this method.

Admittedly, the verification of the properties outlined above is no easy task and often involves a lot of sophisticated mathematics. However, this approach is capable of solving problems — in particular, noncoercive ones — which are out of reach for the other known techniques. We refer to (the introduction of) [3], as well as to the references given there, for some applications to parabolic problems not being reaction-diffusion systems. More recent applications concern the Navier-Stokes equations ([10], [13], [14]), non-Newtonian fluids ([4], [5], [7]), coupled systems of countably and uncountably many reaction-diffusion equations arising from coagulation-fragmentation models ([9], [17]), parabolic equations involving measures ([12], [15], [16], [33], [34]), semilinear parabolic equations with nonlinearities of critical growth ([16], [18], [19]), and nonlinear dynamical boundary conditions [20].

9. A broader view

In order to summarize and to give a somewhat broader view I discuss now some of the interrelationships of the theory of parabolic evolution equations with other fields of analysis, as indicated in the following diagram



I have put parabolic evolution equations in the middle, since they are in the center of my present interest, and have grouped around them several other subjects. I did not put arrows on the connecting lines since in many cases the interaction is bilateral.

Let us start at the left upper corner. It is well-known — and I have taken reaction-diffusion systems as an example — that many concrete models for the understanding of phenomena in science lead to parabolic evolution equations and, vice versa, results on parabolic evolution equations have immediate interpretations and consequences for those models.

Partial differential equations are, of course, intimately connected with parabolic evolution equations. However, neither forms a subfield of the other. For example, parabolic evolution equations encompass also other systems like integro-differential equations or infinite systems of reaction-diffusion equations involving even uncountably many unknowns, as they occur in statistical physics (see [9], [17]).

The connection between parabolic evolution equations and Fourier analysis lies on a more technical level and can be described adequately by more detailed explanations only.

As pointed out earlier, the choice of the correct state space is fundamental when studying partial differential equations, parabolic evolution equations in particular. The well-developed theory of function spaces provides us with a wide variety of possibilities. Spaces more refined than integer order Sobolev spaces like Besov and Bessel potential spaces have become increasingly important during the last years. This is true, in particular, in the study of the Navier-Stokes equations (cf. [10], [13], [14], and the references therein).

Interpolation theory provides us, on a more abstract level, with the right tool for measuring very precisely regularity properties which are the key to a successful approach to nonlinear equations.

As I have explained, semigroup theory is precisely what is needed — on the abstract level — to derive the most general local existence theory for parabolic evolution equations. Spectral theory comes in when one starts to study stability questions and the long-time behavior.

I could not into detail on the relation between parabolic evolution equations and infinite-dimensional harmonic analysis. Among other things, it has to do with "maximal regularity" questions and has, in particular during the last few years, stimulated much research in Banach space theory. I only want to mention recent results of Weis [36], Kalton [28], and others on Fourier multiplier theorems with operator-valued symbols in vector-valued L_p-spaces. Those results are tied to the theory of UMD spaces, for example. (Also see [22], [23] and, for results valid for arbitrary Banach spaces, [8], [11].)

Finally, methods from nonlinear functional analysis, fixed point theorems, bifurcation theory, etc., play an important rôle in the difficult and fascinating investigation of qualitative properties of the semiflows generated by parabolic evolution equations.

I hope that this enumeration of subjects, which is far from being complete, shows that the field of parabolic evolution equations is a fascinating one, invoking a lot of deep and beautiful mathematics.

10. Future directions

I close this article by a word on possible future directions of research. In order not to get lost in general speculations, I mention one particular aspect only, namely the fact that *systems* of evolution equations are of predominant interest for applications. This is true, in particular, for strongly coupled systems of quasilinear reaction-diffusion equations. In this case, as indicated in the preceding sections, the basic theory on local existence, uniqueness, and continuous dependence on the data is now available. Thus it is time to concentrate on the study of the qualitative behavior of the associated semiflow.

Admittedly, this is a formidable task since most of the known tools, which have been used very successfully in the study of a single parabolic equation, do not work in this case. In particular:

- In most systems of interest there is no variational structure for the principal part. This implies that energy estimates are not available, in general.

- The maximum principle does not hold for systems. That principle is — implicitly or explicitly — one of the main tools for the qualitative study of a single second order parabolic equation.

The nonavailability of these basic tools makes these problems extremely difficult. However, it is a great challenge and fascinating task to develop new techniques and new ideas.

Clearly, there is no recipe for finding new results and having new ideas. Nevertheless, David Hilbert has a worthwhile advice for this case also. In his Paris speech of the year 1900 he says:

Wenn uns die Beantwortung eines mathematischen Problems nicht gelingen will, so liegt häufig der Grund darin, daß wir noch nicht den allgemeineren Gesichtspunkt erkannt haben, von dem aus das vorgelegte Problem nur als einzelnes Glied einer Kette verwandter Probleme erscheint. Nach Auffindung dieses Gesichtspunktes wird häufig nicht nur das vorgelegte Problem unserer Erforschung zugänglicher, sondern wir gelangen so zugleich in den Besitz einer Methode, die auf die verwandten Probleme anwendbar ist…
Eine noch wichtigere Rolle als das Verallgemeinern spielt — wie ich glaube — bei der Beschäftigung mit mathematischen Problemen das Spezialisieren. Vielleicht in den meisten Fällen, wo wir die Antwort auf eine Frage vergeblich suchen, liegt die Ursache des Mißlingens darin, daß wir einfachere und leichtere Probleme als das vorgelegte noch nicht oder noch unvollkommen erledigt haben. Es kommt dann alles darauf an, diese leichteren Probleme aufzufinden und ihre Lösung mit möglichst vollkommenen Hilfsmitteln und durch verallgemeinerungsfähige Begriffe zu bewerkstelligen.[2]

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[2] If we do not succeed in solving a mathematical problem then, very often, the reason is that we did not yet discover the more general point of view from which the given problem appears to be a link in a chain of related problems. Having found this point of view, not only the given problem becomes more accessible to our research, but we also gain a method which is applicable to related problems…

In dealing with mathematical problems, specializing plays — as I believe — an even more important rôle than generalizing. Perhaps in most cases in which we fail to find an answer, the reason for this failure is that we did not solve, at least not completely, problems simpler and easier than the given one. Everything amounts to finding these easier problems and to solve them by using tools which are as perfect as possible and concepts which are fit to be generalized.

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Herbert Amann
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH–8057 Zürich
Switzerland
amann@math.unizh.ch