Function Spaces on Uniformly Regular and Singular Riemannian Manifolds

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In memoriam Giuseppe Da Prato
who was a trailblazer of abstract evolution equations
and maximal regularity

Abstract. This paper shows that the basic properties of Sobolev, Besov, and Bessel potential spaces are valid on Riemannian manifolds with boundary, which either have bounded geometry or posses singularities. In the latter case the appropriate setting is that of Kondratiev-type weighted spaces. The importance and usefulness of our results are indicated by a demonstration of a maximal regularity result for a linear parabolic initial value problem on singular manifolds.

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1. Introduction

It is well-known that Banach spaces of distributions—most notably Sobolev and Hölder spaces—play a decisive role in the study of linear and nonlinear differential equations. Wheras the theory of function spaces on subdomains of Euclidean spaces is well developed, this is by far not true if the underlying domain is a Riemannian manifold.

In recent years, the theory of differential equations on Riemannian manifolds has found increasing interest. This is motivated both by intrinsic differential geometric questions and by problems from applied fields like fluid mechanics or numerical analysis, for example. In those connections evolution equations of parabolic type are of predominant importance.

During the last decades, the theory of linear and nonlinear parabolic evolution equations on general Banach spaces has made great progress. In particular, the local theory of quasilinear parabolic equations is by now well established. It is based on linearization and so-called maximal regularity results for linear equations (see, for example, [3], [50, Chapter 5]).

In order that these abstract results become available for the study of parabolic evolution equations in the global analysis setting, a good understanding of embedding, interpolation, point-wise multiplications, and trace properties of Banach spaces of functions—more generally, of sections of vector bundles—is fundamental. It is the purpose of this paper to provide such results.

Basically, our paper consists of three parts. In the first one, which comprises Sections 2–4, it is shown that, on manifolds with boundary and bounded geometry, Sobolev, Besov, and Bessel potential spaces are defined and possess the same properties as in the well-known Euclidean setting.

In the second part we consider a class of singular manifolds. In this frame weighted spaces of Kondratiev-type occur naturally. By an easy transposition technique we prove that these spaces too possess all the properties known to hold for the classical unweighted spaces. This is done in Sections 5 and 6. Then, in Section 7, we demonstrate, by a prototypical example, how the transposition method yields maximal regularity results for uniformly parabolic equations on singular manifolds.

It remains to produce concrete classes of singular manifolds. This is achieved in the third part, in Sections 8–10.

In order to keep the exposition simple we present our results for function spaces only, although everything applies to spaces of sections of general tensor bundles. Also, as for applications, we consider merely singular manifolds with smooth cuspidal point singularities. Cuspidal wedges and corners are treated in [12] where an elaborate exposition of the theory is provided. For this reason the proofs in the present paper are rather brief and sometimes sketchy only.

So as to make this paper accessible to a broad readership, in the Appendix we have collected the notations and conventions which we use throughout the main body of this paper without further ado.

2. Bounded Geometry

Let (M,g) be an m-dimensional, $m \geq 1$, Riemannian manifold with (possibly empty) boundary ∂M. For each $p \in \mathring{M} := M \setminus \partial M$ and v in T_pM there exist a maximal open interval $J_p(v)$ about 0 in $\mathbb{R}$ and a unique geodesic $\gamma_p(\cdot,v)\colon J_p(v) \to M$ satisfying $\gamma_p(0,v) = p$ and $\dot\gamma_p(0,v) = v$, the maximal geodesic 'starting at p in direction v'. The exponential map at p is defined by $\exp_p(v) := \gamma_p(1,v)$ for all $v \in T_pM$ with $1 \in J_p(v)$. Given any $p \in \mathring{M}$, there exists $\rho(p) > 0$ such that $\exp_p$ is a diffeomorphism from the open ball about the origin in T_pM with radius $\rho(p)$ onto an open neighborhood of p in $\mathring{M}$. The supremum of all such $\rho(p)$ is the injectivity radius, $\mathrm{inj}(p)$, of $\mathring{M}$ at p. Given a nonempty subset S of $\mathring{M}$,

$$\mathrm{inj}(S) := \inf_{p\in S} \mathrm{inj}(p)$$

is the injectivity radius of S. Note that $\mathrm{inj}(\mathring{M})$ may be zero.

Suppose $\partial M(\varepsilon)$ is an open neighborhood of ∂M in M and χ is a diffeomorphism from $\partial M(\varepsilon)$ onto $\partial M \times [0,\varepsilon)$ such that $\chi(p) = (p,0)$ for $p \in \partial M$. Then $\big(\partial M(\varepsilon), \chi\big)$ is a uniform collar of ∂M in M of width ε. It is a *geodesic collar* if

$$\chi^{-1}(p,t) = \gamma_p\big(t,\nu(p)\big), \ (p,t) \in \partial M \times [0,\varepsilon), \tag{2.1}$$

where ν is the inner (unit) normal of ∂M. Lastly, (M,g) has *bounded curvature* if all covariant derivatives of the Riemannian curvature tensor are bounded.

Definition 2.1. A Riemannian manifold (M,g) without boundary is said to have *bounded geometry* if it has a positive injectivity radius and bounded curvature. If $\partial M \neq \emptyset$, then (M,g) *has bounded geometry* if

(i) ∂M has a uniform geodesic collar of width ε in M.
(ii) If $0 < r < \varepsilon$, then $\mathrm{inj}\big(M \setminus \partial M(r)\big) > 0$.
(iii) $(\partial M, g_{\partial M})$ has bounded geometry.
(iv) (M,g) has bounded curvature.

Here $g_{\partial M}$ is the restriction of g to the subbundle $T\partial M$ of TM. □

Assume $\partial M \neq \emptyset$ and let $\nabla_{g_{\partial M}}$ be the Levi–Civita covariant derivative of $(\partial M, g_{\partial M})$. Th. Schick [53] defines that (M,g) has bounded geometry if conditions (i), (ii), and (iv) of Definition 2.1 apply, $(\partial M, g_{\partial M})$ has a positive injectivity radius, and all $\nabla_{g_{\partial M}}$ covariant derivatives of the second fundamental form are bounded. (Also see B. Ammann, N. Große, and V. Nistor [15] for a variant.)

Theorem 2.2. *Assume $\partial M \neq \emptyset$. Then (M,g) has bounded geometry iff it has bounded geometry in the sense of Schick.*

Proof. [12, Theorem XI.2.4.11]. □

3. Uniform Regularity

To obtain flexible local descriptions of Riemannian manifolds with bounded geometry we introduce the concept of uniformly regular Riemannian manifolds.

Let $U_\kappa = \mathrm{dom}(\kappa)$ be the coordinate patch of a local chart κ for M. Then κ is *normalized*, provided

$$\kappa(U_\kappa) = Q_\kappa^m := \begin{cases} (-1,1)^m, & \text{if } U_\kappa \subset \mathring{M}, \\ [0,1) \times (-1,1)^{m-1}, & \text{if } U_\kappa \cap \partial M \neq \emptyset. \end{cases} \tag{3.1}$$

An atlas $\mathfrak{K}$ is *normalized* if it consists of normalized charts. A normalized atlas is *shrinkable* if there exists $r \in (0,1)$ such that $\{\, \kappa^{-1}(rQ_\kappa^m) \ ;\ \kappa \in \mathfrak{K} \,\}$ is a covering of M. It has *finite multiplicity* if there is $k \in \mathbb{N}$ such that any intersection of more than k coordinate patches is empty.

Definition 3.1. An atlas $\mathfrak{K}$ for M is *uniformly regular* (ur) if

(i) it is normalized, shrinkable, and has finite multiplicity.
(ii) $\widetilde{\kappa} \circ \kappa^{-1} \in BUC^\infty\big(\kappa(U_{\kappa\widetilde{\kappa}}), \mathbb{R}^m\big)$ and
$\|\widetilde{\kappa} \circ \kappa^{-1}\|_{k,\infty} \le c(k)$, $\kappa, \widetilde{\kappa} \in \mathfrak{K}$, $k \in \mathbb{N}$. □

Here $U_{\kappa\widetilde{\kappa}} := U_\kappa \cap U_{\widetilde{\kappa}}$ is understood to be nonempty, and $\|\cdot\|_{k,\infty}$ is the norm in BUC^k, the space of bounded and uniformly continuous C^k functions. Since M is separable and metrizable, it is not difficult to see that a ur atlas is countable, that is, finite or countably infinite. If $\mathfrak{K}$ and $\widetilde{\mathfrak{K}}$ are atlases, $\mathfrak{N}(\kappa, \widetilde{\mathfrak{K}}) := \{\, \widetilde{\kappa} \in \widetilde{\mathfrak{K}} \ ;\ U_{\kappa\widetilde{\kappa}} \ne \emptyset \,\}$, $\kappa \in \mathfrak{K}$.

Definition 3.2. Two ur atlases $\mathfrak{K}$ and $\widetilde{\mathfrak{K}}$ are *equivalent* if

(i) $\operatorname{card} \mathfrak{N}(\kappa, \widetilde{\mathfrak{K}}) + \operatorname{card} \mathfrak{N}(\widetilde{\kappa}, \mathfrak{K}) \le c$, $\kappa \in \mathfrak{K}$, $\widetilde{\kappa} \in \widetilde{\mathfrak{K}}$.
(ii) $\widetilde{\kappa} \circ \kappa^{-1} \in BUC^\infty\big(\kappa(U_{\kappa\widetilde{\kappa}}), \mathbb{R}^m\big)$ with
$\|\widetilde{\kappa} \circ \kappa^{-1}\|_{k,\infty} + \|\kappa \circ \widetilde{\kappa}^{-1}\|_{k,\infty} \le c(k)$, $\kappa \in \mathfrak{K}$, $\widetilde{\kappa} \in \widetilde{\mathfrak{K}}$, $k \in \mathbb{N}$. □

This induces an equivalence relation in the class of all ur atlases. Any equivalence class is a *ur structure* for M. A *ur manifold* is a manifold M together with a ur structure. Then $\mathfrak{K}$ is a *ur atlas for M* iff it belongs to this structure.

Definition 3.3. (M, g) is said to be a *uniformly regular Riemannian (urR) manifold* if there exists a ur atlas $\mathfrak{K}$ for M such that

(i) $\kappa_* g \sim g_m$, $\kappa \in \mathfrak{K}$,
(ii) $\|\kappa_* g\|_{k,\infty} \le c(k)$, $\kappa \in \mathfrak{K}$, $k \in \mathbb{N}$. □

Here $\kappa_* g$ is the local representation of g on TQ_κ^m, the push-forward by κ. Furthermore, $\sim$ holds uniformly w.r.t. $\kappa \in \mathfrak{K}$. It is immediate by Definition 3.2 that this determination is independent of the specific ur atlas $\mathfrak{K}$.

We present a short list of easy examples on which we build below. More sophisticated urR manifolds are found in Sections 8–10.

Examples 3.4. (a) $(\mathbb{R}^m, g_m)$ and $(\mathbb{H}^m, g_m)$, where $\mathbb{H}^m := \mathbb{R}_+ \times \mathbb{R}^{m-1}$, are urR manifolds.

(b) Compact manifolds are ur and all ur Riemannian metrics thereon are equivalent.

(c) Suppose that (M_i, g_i), $i = 1, 2$, are ur and either ∂M_1 or ∂M_2 is empty. Then $(M_1 \times M_2,\ g_1 + g_2)$ is a urR manifold. Otherwise, it is a urR manifold with corners.

(d) Let $f \colon (M_1, g_1) \to (M_2, g_2)$ be an isometric diffeomorphism between Riemannian manifolds. Then (M_1, g_1) is ur iff (M_2, g_2) is so. An atlas $\mathfrak{K}$ for M_1 is ur iff $f_*\mathfrak{K} := \{\, f_*\kappa \ ;\ \kappa \in \mathfrak{K} \,\}$ is a ur atlas for M_2. □

The following fundamental result shows that urR manifolds yield local descriptions of manifolds with bounded geometry.

Theorem 3.5. *A Riemannian manifold is ur iff it has bounded geometry.*

The concept of urR manifolds has been introduced by the author in [4]. In that paper it has been observed that a manifold without boundary and bounded geometry is ur. The converse is due to M. Disconzi, Y. Shao, and G. Simonett [33]. If $\partial M \neq \emptyset$, then Theorem 3.5 is proved in [12, Theorems XI.2.4.1 and XI.2.4.8].

Let (M, g) be ur and $\varepsilon > 0$. It is not difficult to see that there exists a ur atlas $\mathfrak{K}$ such that $\operatorname{diam}_g(U_\kappa) < \varepsilon$ for $\kappa \in \mathfrak{K}$.

An important technical property of ur manifolds is the subsequent fact. We denote by $\mathcal{D}$ the space of smooth functions with compact support.

Lemma 3.6. *Let $\mathfrak{K}$ be a ur atlas. There exist $\pi_\kappa \in \mathcal{D}\big(U_\kappa, [0,1]\big)$, $\kappa \in \mathfrak{K}$, such that*

(i) $\{\, \pi_\kappa^2 \ ;\ \kappa \in \mathfrak{K} \,\}$ *is a partition of unity subordinate to the open covering* $\{\, U_\kappa \ ;\ \kappa \in \mathfrak{K} \,\}$ *of* M.
(ii) $\|\kappa_* \pi_\kappa\|_{k,\infty} \leq c(k)$, $\kappa \in \mathfrak{K}$, $k \in \mathbb{N}$.
(iii) $\chi \in \mathcal{D}\big((-1,1)^m, [0,1]\big)$ *and* $\chi | \operatorname{supp}(\kappa_* \pi_\kappa) = 1$ *for* $\kappa \in \mathfrak{K}$.

Proof. [4, Lemma 3.2]. □

The family $\big\{\, (\pi_\kappa, \chi) \ ;\ \kappa \in \mathfrak{K} \,\big\}$ is said to be a *localization system subordinate* to $\mathfrak{K}$.

4. Function Spaces

In this section we introduce the most important Banach spaces of distributions on urR manifolds and discuss their main properties.

Let (M, g) be a Riemannian manifold, $k \in \mathbb{N}$, and $1 \leq p \leq \infty$. Then

$$\|u\|_{k,p} = \|u\|_{k,p,M} := \sum_{j=0}^{k} \big\| \, |\nabla^k u|_{g_0^k} \big\|_{L_p(M)}, \quad u \in C^k(M), \tag{4.1}$$

where $\nabla = \nabla_g$ denotes the Levi–Civita covariant derivative and $L_p(M)$ equals $L_p(M, dV_g)$.

If $p = q < \infty$, then the *Sobolev space* $W_q^k(M)$, $1 \leq q < \infty$, is the completion of $\mathcal{D}(M)$ in $L_{1,\mathrm{loc}}(M)$ with respect to the norm $\|\cdot\|_{k,q}$. The *space* $BC^k(M)$ *of bounded continuous* C^k *functions* on M is the Banach space of all $u \in C^k(M)$ satisfying $\|u\|_{k,\infty} < \infty$, endowed with the norm $\|\cdot\|_{k,\infty}$. It follows that $W_q^0(M)$ equals $L_q(M)$ and $BC^0(M) = BC(M)$.

We write $(\cdot,\cdot)_{\theta,p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, resp. $[\cdot,\cdot]_\theta$, $0 < \theta < 1$, for the real, resp. complex, interpolation functors.

The *Besov space* $B^s_{p,r}(M)$ is defined for $s > 0$ and $1 \le p, r \le \infty$ by

$$B^s_{q,r}(M) := \begin{cases} \bigl(W^k_q(M), W^{k+1}_q(M)\bigr)_{s-k,\,r}, & \text{if } k < s < k+1, \\ \bigl(W^k_q(M), W^{k+2}_q(M)\bigr)_{1/2,\,r}, & \text{if } s = k+1, \end{cases} \tag{4.2}$$

if $p = q < \infty$, and by

$$B^s_{\infty,r}(M) := \begin{cases} \bigl(BC^k(M), BC^{k+1}(M)\bigr)_{s-k,\,r}, & \text{if } k < s < k+1, \\ \bigl(BC^k(M), BC^{k+2}(M)\bigr)_{1/2,\,r}, & \text{if } s = k+1, \end{cases}$$

where $k \in \mathbb{N}$.

We introduce *Bessel potential spaces* $H^s_q(M)$ for $s \ge 0$ and $1 < q < \infty$ by

$$H^s_q(M) := \begin{cases} \bigl[W^k_q(M), W^{k+1}_q(M)\bigr]_{s-k}, & \text{if } k < s < k+1, \\ W^k_q(M), & \text{if } s = k. \end{cases} \tag{4.3}$$

As usual, $B^s_p(M) := B^s_{p,p}(M),\ \ 1 \le p \le \infty$. Then *Slobodeckii spaces* are specified by

$$W^s_q(M) := B^s_q(M),\ s \in \mathbb{R}_+ \setminus \mathbb{N},\ 1 \le q < \infty,$$

and $\bigl[\,W^s_q(M)\ ;\ s \ge 0\bigr]$ is the *Sobolev–Slobodeckii space scale.* Also,

$$BC^s(M) := B^s_\infty(M),\ s \in \mathbb{R}_+ \setminus \mathbb{N},$$

are the *Hölder spaces*, and $\bigl[\,BC^s(M)\ ;\ s \ge 0\bigr]$ is the *Hölder space scale.* Note, however, that $B^k_\infty(M) \ne BC^k(M)$ if $k \in \mathbb{N}$.

Remarks 4.1. (a) For the sake of easy presentation we restrict ourselves to function spaces. However, everything said in this paper applies to spaces of sections of tensor bundles $(T^\sigma_\tau M, g^\tau_\sigma),\ \ \sigma, \tau \in \mathbb{N}$. For this it suffices to replace (4.1) by

$$\|u\|_{k,p,T^\sigma_\tau M} := \sum_{j=0}^k \bigl\|\, |\nabla^j u|_{g^{\tau+j}_\sigma} \bigr\|_{L_p(M)},\ u \in C^k(T^\sigma_\tau M). \tag{4.4}$$

The preceding procedures then lead to the Sobolev spaces $W^k_q(T^\sigma_\tau M)$, the spaces $BC^k(T^\sigma_\tau M)$, and, consequently, to $B^s_{p,r}(T^\sigma_\tau M)$ and $H^s_q(T^\sigma_\tau M)$.

(b) It is clear from (4.4) that

$$\nabla \in \mathcal{L}\bigl(\mathfrak{F}^{k+1}(T^\sigma_\tau M), \mathfrak{F}^k(T^\sigma_{\tau+1} M)\bigr),\ \mathfrak{F}^k \in \{W^k_q, BC^k\}.$$

Hence interpolation and (4.2), (4.3) yield

$$\nabla \in \mathcal{L}\bigl(\mathfrak{F}^{s+1}(T^\sigma_\tau M), \mathfrak{F}^s(T^\sigma_{\tau+1} M)\bigr)$$

if $\mathfrak{F}^s$ belongs to

$$\{\, W^s_q\ ;\ 1 \le q < \infty \,\} \cup \{\, H^s_q\ ;\ 1 < q < \infty \,\} \cup \{\, B^s_{p,r}\ ;\ 1 \le p, r \le \infty \,\}. \qquad \square$$

It should be noted that these definitions apply to any Riemannian manifold. However, in such a generality these spaces are not too useful since they may not possess important embedding and interpolation properties (e.g., [41]). The situation is different for urR manifolds. This is due to the fundamental retraction-coretraction theorem below. Thus we now suppose that

- (M,g) is a urR manifold.

Given a nonempty set S, $\Gamma(S) := \mathbb{R}^S$. Let $\mathfrak{K}$ be a ur atlas. We introduce

$$\mathbb{M}_\kappa := \begin{cases} \mathbb{R}^m, & \text{if } U_\kappa \subset \mathring{M}, \\ \mathbb{H}^m, & \text{otherwise}, \end{cases}$$

and

$$\boldsymbol{\Gamma}(\mathbb{M}) := \prod_{\kappa\in\mathfrak{K}} \Gamma(\mathbb{M}_\kappa).$$

Suppose that $\{ (\pi_\kappa, \chi) \ ; \ \kappa \in \mathfrak{K} \}$ is a localization system subordinate to $\mathfrak{K}$. We set

$$\mathfrak{r}_\kappa^c u := \kappa_*(\pi_\kappa u),\ u \in \Gamma(M), \quad \mathfrak{r}_\kappa v := \pi_\kappa \kappa^*(\chi v),\ v \in \Gamma(\mathbb{M}_\kappa).$$

Then

$$\mathcal{R}^c \colon \Gamma(M) \to \boldsymbol{\Gamma}(\mathbb{M}),\ u \mapsto (\mathfrak{r}_\kappa^c u) \tag{4.5}$$

and

$$\mathcal{R} \colon \boldsymbol{\Gamma}(\mathbb{M}) \to \Gamma(M),\ (v_\kappa) \mapsto \textstyle\sum_{\kappa\in\mathfrak{K}} \mathfrak{r}_\kappa v_\kappa \tag{4.6}$$

are linear mappings, and $\mathcal{R}^c$ is a right inverse of $\mathcal{R}$.

If $\mathfrak{F}(\mathbb{M}_\kappa)$ is a linear subspace of $\Gamma(\mathbb{M}_\kappa)$ 'of the same type for each κ', then

$$\boldsymbol{\mathfrak{F}}(\mathbb{M}) := \prod_\kappa \mathfrak{F}(\mathbb{M}_\kappa) \subset \boldsymbol{\Gamma}(\mathbb{M}).$$

For example, $\boldsymbol{C}^k(\mathbb{M}) = \prod_\kappa C^k(\mathbb{M}_\kappa)$.

Let X and Y be Banach spaces and $r \in \mathcal{L}(X,Y)$. Then r is a *retraction* from X onto Y if it has a continuous right inverse r^c. In this case (r, r^c) is called r-c pair for (X,Y).

To have a unified presentation, we suppose that the symbol $\mathcal{X}^s$ belongs to one of the following sets of symbols, where BUC means bounded and uniformly continuous.

$$\{ BUC^k \ ; \ k \in \mathbb{N} \}, \qquad \{ W_q^s \ ; \ 1 \le q < \infty,\ s \ge 0 \},$$
$$\{ H_q^s \ ; \ 1 < q < \infty,\ s \ge 0 \}, \quad \{ B_{p,r}^s \ ; \ 1 \le p, r \le \infty,\ s > 0 \},$$

where $s = k$ in the first case. The Banach spaces $\mathcal{X}^s(\mathbb{M}_\kappa)$ are particular instances of the (much more general) spaces studied in great detail in [8].

We put, for $\boldsymbol{v} = (v_\kappa) \in \boldsymbol{\mathcal{X}}^s(\mathbb{M})$,

$$\|\boldsymbol{v}\|_{\ell_p(\boldsymbol{\mathcal{X}}^s(\mathbb{M}))} := \big\| \big(\|v_\kappa\|_{\mathcal{X}^s(\mathbb{M}_\kappa)}\big) \big\|_{\ell_p},\ 1 \le p \le \infty.$$

The linear subspace of $\boldsymbol{\mathcal{X}}^s(\mathbb{M})$,

$$\ell_p\big(\boldsymbol{\mathcal{X}}^s(\mathbb{M})\big) := \{ \boldsymbol{v} \in \boldsymbol{\mathcal{X}}^s(\mathbb{M}) \ ; \ \|\boldsymbol{v}\|_{\ell_p(\boldsymbol{\mathcal{X}}^s(\mathbb{M}))} < \infty \},$$

is a Banach space with the norm $\|\cdot\|_{\ell_p(\boldsymbol{\mathcal{X}}^s(\mathbb{M}))}$.

The restriction of $\mathcal{R}$ to $\ell_p\big(\boldsymbol{\mathcal{X}}^s(\mathbb{M})\big)$, resp. of $\mathcal{R}^c$ to $\mathcal{X}^s(M)$, where $\mathcal{X}^k(M)$ equals $BC^k(M)$, is again denoted by $\mathcal{R}$, resp. $\mathcal{R}^c$. Then we can formulate the basic universal r-c theorem.

Theorem 4.2. *Let (M,g) be ur. Then $(\mathcal{R},\mathcal{R}^c)$ is an r-c pair for*

(i) $\big(\ell_\infty(\boldsymbol{BUC}^k(\mathbb{M})), BC^k(M)\big),\ k\in\mathbb{N}$.
(ii) $\big(\ell_\infty(\boldsymbol{B}^s_{\infty,r}(\mathbb{M})), B^s_{\infty,r}(M)\big),\ 1\le r\le\infty,\ s>0$.
(iii) $\big(\ell_q(\boldsymbol{W}^s_q(\mathbb{M})), W^s_q(M)\big),\ 1\le q<\infty,\ s\ge 0$.
(iv) $\big(\ell_q(\boldsymbol{H}^s_q(\mathbb{M})), H^s_q(M)\big),\ 1<q<\infty,\ s\ge 0$.
(v) $\big(\ell_q(\boldsymbol{B}^s_{q,r}(\mathbb{M})), B^s_{q,r}(M)\big),\ 1\le q<\infty,\ 1\le r\le\infty,\ s>0$.

Proof. Cases (i) and (iii) with $s=k$ are established by direct computations. The remaining statements are then obtained by interpolation using the results of [8, Section VI.2]. (The reader may also consult [4] where the basic ideas are already found.) □

Remark 4.3. Since $(\mathcal{R},\mathcal{R}^c)$ is an r-c pair, it is well-known and easy to see that

$$u\mapsto\Big(\sum_\kappa \|\mathfrak{r}^c_\kappa u\|^q_{W^s_q(\mathbb{M}_\kappa)}\Big)^{1/q} \tag{4.7}$$

is a norm for $W^s_q(M)$. Of course, it depends on the choice of the ur atlas $\mathfrak{K}$ and the localization system subordinate to it. However, since $W^s_q(M)$ has been globally defined, another choice of a ur atlas and a corresponding localization system yields an equivalent norm. Analogous observations apply to the spaces $BC^k(M)$, $H^s_q(M)$, and $B^s_{p,r}(M)$. □

One of the fundamental ramifications of the retraction-coretraction theorem is the next statement.

Theorem 4.4. *Let (M,g) be ur. The spaces $BC^k(M)$, $W^s_q(M)$, $H^s_q(M)$, and $B^s_{p,r}(M)$ enjoy the same density, embedding, and interpolation properties as the corresponding classical spaces on $(\mathbb{R}^m,g_m)$, resp. $(\mathbb{H}^m,g_m)$.*

Proof. This follows from the properties of r-c pairs, the features of the associated function spaces on $\mathbb{M}_\kappa$, and the properties of Banach-space-valued sequence spaces. □

We content ourselves by presenting just one of the many consequences, namely, Sobolev-type embedding theorems.

Theorem 4.5. *Assume that (M,g) is ur.*

(i) (Sobolev) *Suppose that $q_0,q_1\in[1,\infty)$ and $0\le s_0<s_1$ satisfy*

$$s_1-m/q_1\ge s_0-m/q_0$$

with a strict inequality unless $s_0,s_1\in\mathbb{N}$. Then $W^{s_1}_{q_1}(M)\overset{d}{\hookrightarrow}W^{s_0}_{q_0}(M)$.

(ii) (Morrey) *If $0\le t<s-m/q$, then $W^s_q(M)\hookrightarrow BC^t(M)$.*

(iii) (Gagliardo–Nirenberg) *Assume that* $0 \le s_0 < s < s_1,\ 1 \le q, q_0, q_1 < \infty$, *and* $0 < \theta < 1$. *Set*

$$s_\theta := (1-\theta)s_0 + \theta s_1, \quad 1/q(\theta) := (1-\theta)/q_0 + \theta/q_1.$$

Let

$$s - m/q = s_\theta - m/q(\theta), \quad q \ge q(\theta),$$

where $q_i > 1$ *if* $s_i \in \mathbb{N}$. *Then* $W_{q_0}^{s_0}(M) \cap W_{q_1}^{s_1}(M) \hookrightarrow W_q^s(M)$ *and*

$$\|u\|_{s,q} \le c\,\|u\|_{s_0,q_0}^{1-\theta}\,\|u\|_{s_1,q_1}^{\theta}.$$

Proof. [12, Subsection XII.3.3]. □

With the help of the r-c pair $(\mathcal{R}, \mathcal{R}^c)$ the point-wise multiplier theorems of [8, Section VII.6] can also be lifted to the manifold (M, g). We restrict ourselves to present the most important of such results.

Let $\mathfrak{F}_i(M),\ \ i = 0, 1, 2$, be Banach spaces of regular distributions, that is, $\mathfrak{F}_i(M) \hookrightarrow L_{1,\mathrm{loc}}(M)$. We write

$$\mathfrak{F}_0(M) \bullet \mathfrak{F}_1(M) \hookrightarrow \mathfrak{F}_2(M)$$

if the point-wise product $(u_0, u_1) \mapsto u_0 \bullet u_1$ restricts to a continuous bilinear map from $\mathfrak{F}_0(M) \times \mathfrak{F}_1(M)$ into $\mathfrak{F}_2(M)$.

Theorem 4.6. *Suppose that* $s, t \ge 0$. *Then*

$$BC^s(M) \bullet W_q^t(M) \hookrightarrow W_q^t(M),\ 1 \le q < \infty, \tag{4.8}$$

provided either $s > t$ *or* $s = t \in \mathbb{N}$, *and*

$$BC^s(M) \bullet BC^t(M) \hookrightarrow BC^t(M),\ 0 \le t \le s. \tag{4.9}$$

If $s > m/q$, *then* $W_q^s(M)$ *is a multiplication algebra.*

Assume $\partial M \ne \emptyset$. If (M, g) is ur, then so is $(\partial M, g_{\partial M})$. Hence the spaces $BC^k(\partial M)$, $W_q^s(\partial M)$, $H_q^s(\partial M)$, and $B_{p,r}^s(\partial M)$ are defined and the analogue of Theorem 4.4 applies.

Let $\big(\partial M(\varepsilon), \chi\big)$ be a geodesic collar of ∂M. The normal derivative of order $j \in \mathbb{N}$ of $u \in C^\infty(M)$ at $p \in \partial M$ is defined by, see (2.1),

$$\partial_\nu^j u(p) = \frac{\partial^j u}{\partial \nu^j}(p) := \Big(\frac{d}{dt}\Big)^j (u \circ \gamma_p)(p, t)\Big|_{t=0}.$$

Thus $\partial_\nu^0 u = u|\partial M$, the *trace* of u on ∂M.

The following *trace theorem* is of predominant importance in the theory of boundary value problems. It also explains why Besov spaces, Slobodeckii spaces in particular, are of outstanding significance.

Theorem 4.7. *Suppose that* (M, g) *is ur and* $\partial M \ne \emptyset$. *Also suppose that*

$$1 < q < \infty,\ j \in \mathbb{N},\ s > j + 1/q.$$

Then $\vec{\partial}_\nu^j := (\partial_\nu^0, \ldots, \partial_\nu^j)$ *is a retraction from* $W_q^s(M)$ *and from* $H_q^s(M)$ *onto*

$$\prod_{i=0}^{j} B_q^{s-i-1/q}(\partial M).$$

Proof. Let $\mathcal{X}^t$ be one of the symbols W_q^t or H_q^t. Denote by $(\dot{\mathcal{R}}, \dot{\mathcal{R}}^c)$ the restriction of $(\mathcal{R}, \mathcal{R}^c)$ to ∂M (in the obvious sense). It is seen that $(\dot{\mathcal{R}}, \dot{\mathcal{R}}^c)$ is an r-c pair for

$$\big(\ell_q(\boldsymbol{\mathcal{X}}^t(\mathbb{R}^{m-1})), \mathcal{X}^t(\partial M)\big).$$

Using it, we carry the (isotropic) half-space trace Theorem VIII.1.3.2 of [8] over to ∂M. □

Remarks 4.8. (a) Suppose we consider the more general case of tensor-valued sections (Remark 4.1(a)). Then the spaces $\mathcal{X}^s(\mathbb{M}_\kappa)$ in Theorem 4.2 have to be replaced by $\mathcal{X}^s(\mathbb{M}_\kappa, T_\tau^\sigma \mathbb{R}^m)$. Also, (4.8) in the point-wise multiplier theorem now reads

$$BC^s(T_0^j M) \bullet W_q^t(T_{\tau+j}^\sigma M) \hookrightarrow W_q^t(T_\tau^\sigma M)$$

for $\sigma, \tau, j \in \mathbb{N}$. An analogous statement holds for (4.9). With these modifications everything said above applies to the spaces of T_τ^σ-sections.

In fact, in [12] we admit general metric vector bundles (possibly with Banach space fibers) equipped with a ur metric connection. We also mention M. Kohr and V. Nistor [43] for some results on Sobolev spaces on general Riemannian manifolds.

(b) If $\mathring{W}_q^s(M)$ is the closure of $\mathcal{D}(\mathring{M})$ in $W_q^s(M)$, then $\mathring{W}_q^s(M) = W_q^s(M)$ for $0 \le s < 1/q$, and

$$\mathring{W}_q^s(M) = \ker(\vec{\partial}_q^j), \ \ j + 1/q < s < j + 1 + 1/q, \ \ j \in \mathbb{N}.$$

(c) Sobolev spaces of negative order are introduced (in the T_τ^σ-setting) by

$$W_q^{-s}(T_\tau^\sigma M) := \big(\mathring{W}_{q'}^s(T_\sigma^\tau M)\big)', \ \ s > 0,$$

w.r.t. the $L_{q'}(T_\sigma^\tau M) \times L_q(T_\tau^\sigma M)$-duality pairing, where $1/q + 1/q' = 1$ and $1 < q < \infty$. Then $(\mathcal{R}, \mathcal{R}^c)$ has a unique extension to these spaces. This easily leads to interpolation and embedding theorems for negative order spaces. □

Nearly all papers on distribution spaces over Riemannian manifolds are concerned with manifolds without boundary. Very often, compactness is additionally required.

Bessel potential spaces have first been introduced by R. Strichartz [60] on complete Riemannian manifolds without boundary as the fractional power spaces of the Laplace–Beltrami operator. Large parts of the theory of Triebel–Lizorkin, Sobolev, Besov, and Bessel potential spaces have been lifted by H. Triebel [61], [62] from $\mathbb{R}^m$ to Riemannian manifolds without boundary and bounded geometry. A unified presentation is given in [63, Chapter 7]. He uses geodesic coordinates to define Triebel-Lizorkin spaces by means of the analog of (4.7). Then Besov spaces are dealt with by real interpolation. For Bessel potential spaces, Strichartz' technique is employed. The restriction to geodesic coordinates does not allow to introduce local coordinate norms on Besov spaces (see the introduction in Subsection 7.3.1 in [63]). N. Große and C. Schneider [37] partly remove this restriction by employing atlases which are uniformly equivalent to geodesic ones. This then leads them to localized norms for Bessel potential and Besov spaces $B_q^s(M),\ 1 < q < \infty$.

In the case of the Hölder–Zygmund spaces $B_\infty^s,\ \ s > 0$, Triebel's approach leads to unnatural restrictions on s which are partially rectified by introducing global Hölder–Zygmund norms based on difference norms along geodesics. This global access is definitely restricted to manifolds without boundaries.

In contrast to all those works, we use a top down approach. This means: the spaces are globally defined and, then, it is shown that they can be localized by the $(\mathcal{R}, \mathcal{R}^c)$ pair. This method has been introduced in [4] in the general context of tensor bundle sections on a class of singular Riemannian manifolds which comprises the family of Riemannian manifolds with bounded geometry having possibly a nonempty boundary. As is witnessed by the results of the present section, this approach is very flexible and allows to transfer practically all results, known to hold in the Euclidean setting, to urR manifolds. In addition, it has the advantage that it automatically guarantees that different choices of the localization ingredients lead to equivalent norms.

Besides its usefulness for lifting the function space results from $\mathbb{R}^m$ to M, the localization technique is fundamental for establishing basic solvability results for (partial) differential equations on urR manifolds (for example [6], [7], [9], [11]. Also see [13] for the earliest implementation of this technique in the simplest case $(\mathbb{R}^m, g_m)$.)

As for Sobolev embedding theorems on Riemannian manifolds: virtually all research concerns the special case

$$W_q^1(M) \hookrightarrow L_{q/(m-q)}(M).$$

This is due to its differential-geometric implications. Particularly noteworthy are the writings of Th. Aubin [18], [19], [20], Th. Aubin and Y.Y. Li [21], E. Hebey [39], [40], E. Hebey and F. Robert [41]. Both Aubin and Hebey also study the problem of optimal embedding constants and consider manifolds of bounded geometry. Supplementary references are E. Hebey and M. Vaugon [42] and O. Druet [34], O. Druet and E. Hebey [35], and H. Cheik Ali [24] as well as the papers cited in these publications. There are many Sobolev estimates on Riemannian manifolds possessing a special geometric structure. A prototypical reference is L. Saloff-Coste [52].

Not much seems to be known about general Sobolev embeddings and Gagliardo–Nirenberg inequalities on noncompact manifolds e.g., L. Adriano and Ch. Xia [1], N. Badr [22]. The situation is different if manifolds with additional geometric structures (for example, curvature bounds, symmetries, etc.) are considered. For this we refer to the differential geometric literature. Also [52] might be of interest.

5. Singular Manifolds

We now introduce a large class of urR manifolds by a conformal change of the metric g with a suitable singularity function. In this section we prove two technical lemmas which are crucial for the following.

Definition 5.1. Let (M,g) be a Riemannian manifold. Assume:

(i) $\rho \in C^\infty\big(M,(0,1]\big)$.
(ii) $\widehat{g} := g/\rho^2$ and $(M,\widehat{g})$ is a urR manifold.
(iii) $\big\| \rho^{k+1} \, |\nabla^k d(\log\rho)|_{g_0^{k+1}} \big\|_{BC(M)} \le c(k)$, $k \in \mathbb{N}$.

Then ρ is said to be a *singularity function* for (M,g), and (M,g,ρ) is a *singular Riemannian manifold* of conformal type, more precisely, of type ρ. □

This is an updated version of the concept of singular manifolds first introduced in [4]. It follows from Corollary 6.2 below that condition (iii) means that ρ is an 'admissible weight' for $\widehat{g}$ in the denomination of B. Ammann, N. Große, and V. Nistor [14].

Remarks 5.2. (a) Definition 5.1 implies that M is ur.

(b) Let ρ_1 and ρ_2 be singularity functions for (M,g) such that $\rho_1 \sim \rho_2$. Then $g/\rho_1 \sim g/\rho_2$. Denoting by $[\![\rho]\!]$ the equivalence class for $\sim$ containing the representative ρ, it is more precise to say that (M,g,ρ) is a *singular manifold of type* $[\![\rho]\!]$, and $[\![\rho]\!]$ is the *singularity type of* M.

(c) Note that $\widehat{g} \sim g$ iff $\rho \sim \mathbf{1}$. If $\rho \not\sim \mathbf{1}$, then there exists a sequence (p_j) which leaves every compact subset of M such that $\big(\rho(p_j)\big)$ converges to 0. Thus ρ captures, in some sense, the singular behavior of (M,g) 'near infinity' (of $(M,\widehat{g})$).

(d) In local coordinates,

$$\widehat{g} = \rho^{-2} g_{ij}\, dx^i \otimes dx^j, \quad \widehat{g}^* = \rho^2 g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

and $\sqrt{\widehat{g}} = \rho^{-m}\sqrt{g}$. Consequently, $|\cdot|_{\widehat{g}^\tau_\sigma} = \rho^{\tau-\sigma}\, |\cdot|_{g^\tau_\sigma}$. □

From now on *our interest focuses on the singular manifold* (M,g,ρ), whereas $(M,\widehat{g})$ is viewed as a 'uniform regularization' (or 'desingularization') thereof. Since $\widehat{g}$ is obtained by a conformal change of g, there is an intimate relationship between (M,g) and $(M,\widehat{g})$. In particular, the distribution spaces on the urR manifold $(M,\widehat{g})$, defined by means of the Levi–Civita covariant derivative $\widehat{\nabla} := \nabla_{\widehat{g}}$, can be expressed solely in terms of $\nabla = \nabla_g$ and ρ. In this section we prepare the necessary technical details.

We set $\mathfrak{F}(\widehat{M}) := \mathfrak{F}(M,\widehat{g})$ for a given Banach space $\mathfrak{F}$ of functions. If $a \in C(T^\sigma_\tau M)$, then (see Remark 4.1(a))

$$\|a\|_{BC(T^\sigma_\tau M)} := \big\| \, |a|_{g^\tau_\sigma} \big\|_{BC(M)}.$$

We write, for $i,j \in \mathbb{N}$,

$$a \in BC^\infty(T^i_j M;\rho) \iff \|\rho^{k+j-i}\nabla^k a\|_{BC(T^i_{j+k}M)} \le c(k), \ k \in \mathbb{N}.$$

Lemma 5.3. *Suppose that $k \geq 1$. There exists*

$$S \in BC^\infty(T_2^1 M; \rho) \tag{5.1}$$

such that

$$\widehat{\nabla} u - \nabla u = S^k u := \sum_{t=1}^{k} \mathsf{C}_{2+t}^1 (S \otimes u), \; u \in C^\infty(T_k^0 M), \; k \geq 1. \tag{5.2}$$

Proof. (1) Let $\mathfrak{K}$ be a ur atlas for M. In local coordinates, $\kappa = (x^1, \ldots, x^m)$,

$$\widehat{\nabla} u - \nabla u = (\widehat{\Gamma}_{ij}^k - \Gamma_{ij}^k) u_k dx^i \otimes dx^j, \quad u = u_k dx^k,$$

with the Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}), \; \partial_i := \frac{\partial}{\partial x^i}.$$

An easy computation shows that

$$\widehat{\Gamma}_{ij}^k - \Gamma_{ij}^k = -\big(\delta_i^k \partial_j + \delta_j^k \partial_i - g_{ij} g^{k\ell} \partial_\ell\big) \log \rho.$$

Using $g^{kj} g_{ji} = \delta_i^k$, we get

$$\delta_i^k \frac{\partial}{\partial x^k} \otimes dx^i = \mathsf{C}_1^2 (g^* \otimes g).$$

Thus, writing $S = S\big(d(\log \rho)\big)$ and

$$S := -2 d(\log \rho) \otimes \mathsf{C}_1^2 (g^* \otimes g) + g \otimes \big(g^\sharp d(\log \rho)\big) \in C^\infty(T_2^1 M),$$

we find

$$\widehat{\nabla} u - \nabla u = S \bullet u, \; u \in C^\infty(T^* M).$$

From $\nabla g = 0$ it follows that

$$\rho^{\ell+1} \nabla^\ell S\big(d(\log \rho)\big) = S\big(\rho^{\ell+1} \nabla^\ell d(\log \rho)\big), \; \ell \in \mathbb{N}. \tag{5.3}$$

Remark 5.2(d) implies $g_{ij} g^{k\ell} = \widehat{g}^{ij} \widehat{g}_{k\ell}$. Hence, by Definition 5.1(ii), $g_{ij} g^{k\ell} \sim 1$ uniformly w.r.t. $\kappa \in \mathfrak{K}$. From this and (5.3) we obtain

$$\|\rho^{\ell+1} \nabla^\ell S\|_{BC(T_{\ell+2}^1 M)} \leq c \, \|\rho^{\ell+1} \nabla^\ell d(\log \rho)\|_{BC(T_{\ell+1}^0 M)}, \; \ell \in \mathbb{N}.$$

This and Definition 5.1(iii) prove that (5.1) holds and that (5.2) applies for $k = 1$ with $S^1 := S$.

(2) The assertion for $k \geq 2$ now follows by a straightforward somewhat tedious calculation. ☐

By means of this lemma we can express $\widehat{\nabla}^k$ in terms of ∇^k, and vice versa:

Lemma 5.4. *Let $k \geq 1$. There exist*

$$a_i^k, b_i^k \in BC^\infty(T_k^i M; \rho), \; 0 \leq i \leq k-1, \tag{5.4}$$

such that

$$\widehat{\nabla}^k u = \nabla^k u + \sum_{i=0}^{k-1} a_i^k \bullet \nabla^i u \tag{5.5}$$

and

$$\nabla^k u = \widehat{\nabla}^k u + \sum_{i=0}^{k-1} b_i^k \bullet \widehat{\nabla}^i u \tag{5.6}$$

for $u \in C^\infty(M)$.

Proof. It is easy to see that there exist

$$a^k \in BC^\infty(T_{k+1}^k M; \rho) \tag{5.7}$$

such that

$$S^k u = a^k \bullet u,\ \ u \in C^\infty(T_k^0 M).$$

We proceed by induction.

(1) If $k = 1$, then $\widehat{\nabla}^1 = \nabla^1 = d$. Hence we set $a_0^1 := 0$. Assume $k = 2$. Lemma 5.3 shows that (5.5) holds with $a_1^2 := S^1$.

(2) Suppose that the assertions pertaining to (5.5) have already been proved for $0 \le j \le k$. Then, by Lemma 5.3,

$$\begin{aligned}
\widehat{\nabla}^{k+1} u &= \widehat{\nabla}\Big(\nabla^k u + \sum_{i=0}^{k-1} a_i^k \bullet \nabla^i u\Big) \\
&= (\nabla + S^k)\Big(\nabla^k u + \sum_{i=0}^{k-1} a_i^k \bullet \nabla^i u\Big) \\
&= \nabla^{k+1} u + \sum_{i=0}^{k-1}\big(\nabla(a_i^k \bullet \nabla^i u) + a^k \bullet a_i^k \bullet \nabla^i u\big) + a^k \bullet \nabla^k u \\
&= \nabla^{k+1} u + \sum_{i=0}^{k-1}(\nabla a_i^k + a^k \bullet a_i^k) \bullet \nabla^i u + \sum_{i=0}^{k-1} a_i^k \bullet \nabla^{i+1} u + a^k \bullet \nabla^k u.
\end{aligned}$$

Set $a_{-1}^k := 0$,

$$a_i^{k+1} := \nabla a_i^k + a^k \bullet a_i^k + a_{i-1}^k,\ \ 0 \le i \le k-1, \tag{5.8}$$

and $a_k^{k+1} := a^k$. Then $\widehat{\nabla}^{k+1} u = \nabla^{k+1} u + \sum_{i=0}^k a_i^{k+1} \bullet \nabla^i u$.

(3) We show, by induction on k, that

$$a_i^k \in BC^\infty(T_k^i M; \rho),\ \ 0 \le i \le k-1. \tag{5.9}$$

If $k = 2$, then $a_0^1 = S^1$ and (5.1) yield the claim.

Assume that (5.9) applies for some $k \ge 2$. Then

$$\rho^{\ell+k+1-i} \nabla^\ell (\nabla a_i^k) = \rho^{\ell+1+k-i} \nabla^{\ell+1} a_i^k,\ \ \ell \in \mathbb{N},$$

and the induction hypothesis guarantee that

$$\|\rho^{\ell+k+1-i} \nabla^\ell (\nabla a_i^k)\|_{BC(T_{k+\ell+1}^i M)} \le c(\ell) \tag{5.10}$$

for $0 \le i \le k-1$. Using the product rule, we find

$$\rho^{\ell+k+1-i} \nabla^\ell (a^k \bullet a_i^k) = \sum_{j=0}^{\ell} \binom{\ell}{j} \rho^{j+1} \nabla^j a^k \bullet \rho^{\ell-j+k-i} \nabla^{\ell-j} a_i^k.$$

From (5.7) it follows

$$\|\rho^{j+1}\nabla^j a^k\|_{BC(T^k_{k+1+j}M)} \le c(\ell),\ 0 \le j \le \ell < \infty.$$

Assumption (5.9) yields

$$\|\rho^{\ell-j+k-i}\nabla^{\ell-j} a^k_i\|_{BC(T^i_{k+\ell-j}M)} \le c(\ell),\ 0 \le j \le \ell,\ 0 \le i \le k-1.$$

This implies

$$\|\rho^{\ell+k+1-i}\nabla^{\ell}(a^k \bullet a^k_i)\|_{BC(T^i_{k+\ell+1}M)} \le c(\ell),\ \ell \in \mathbb{N},\ 0 \le i \le k-1. \tag{5.11}$$

Also, $\rho^{\ell+k+1-i}\nabla^\ell a^k_{i-1} = \rho^{\ell+k-(i-1)}\nabla^\ell a^k_{i-1}$ provides

$$\|\rho^{\ell+k+1-i}\nabla^{\ell} a^k_{i-1}\|_{BC(T^{i-1}_{k+\ell}M)} \le c(\ell),\ \ell \in \mathbb{N},\ 0 \le i \le k-1. \tag{5.12}$$

Thus we obtain from (5.8) and (5.10)–(5.12) that

$$a^{k+1}_i \in BC^\infty(T^i_{k+1}M;\rho),\ 0 \le i \le k-1. \tag{5.13}$$

It follows from $a^{k+1}_k = a^k$ and (5.7) that (5.13) holds also for $i = k$. Thus the induction is complete and (5.9) is established. Due to step (1), this proves the first claim.

(4) Now we look at

$$\widehat{\nabla}^j u = \nabla^j u + \sum_{i=0}^{j-1} a^j_i \bullet \nabla^i u,\ 0 \le j \le k,$$

as a lower triangular system of linear equations in the unknowns $\nabla^i u$ for $0 \le i \le k$. Solving it by forward substitution, we verify that the claim pertaining to (5.6) is also true. □

6. Weighted Spaces

Let $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Given $1 \le q < \infty$, we put

$$\|u\|_{W^{k,\lambda}_q(M;\rho)} := \sum_{i=0}^{k} \big\|\rho^{-\lambda+i-m/q}\,|\nabla^i u|_{g^i_0}\big\|_{L_q(M)}. \tag{6.1}$$

Also

$$\|u\|_{BC^{k,\lambda}(M;\rho)} := \sum_{i=0}^{k} \big\|\rho^{-\lambda+i}\,|\nabla^i u|_{g^i_0}\big\|_{BC(M)}. \tag{6.2}$$

Then the *weighted Sobolev space*

$W^{k,\lambda}_q(M;\rho)$ is the completion of $\big(\mathcal{D}(M), \|\cdot\|_{W^{k,\lambda}_q(M;\rho)}\big)$ in $L_{1,\mathrm{loc}}(M, dV_g)$.

The *weighted space of bounded smooth functions*

$BC^{k,\lambda}(M;\rho)$ is the Banach space of all $u \in C^k(M)$
for which (6.2) is finite, endowed with the norm $\|\cdot\|_{BC^{k,\lambda}(M;\rho)}$.

Weighted Slobodeckii spaces are defined for $1 \le q < \infty$ and $s \in \mathbb{R}_+ \setminus \mathbb{N}$ by

$$W_q^{s,\lambda}(M;\rho) := \bigl(W_q^{k,\lambda}(M;\rho), W_q^{k+1,\lambda}(M;\rho)\bigr)_{s-k,q}, \ k < s < k+1.$$

If we replace $(\cdot,\cdot)_{s-k,q}$ by the complex interpolation functor $[\cdot,\cdot]_{s-k}$, then we obtain the *weighted Bessel potential spaces* $H_q^{s,\lambda}(M;\rho)$, where $H_q^{k,\lambda} := W_q^{k,\lambda}$ for $k \in \mathbb{N}$. *Weighted Besov spaces* are defined analogously to (4.2). This yields, in particular, weighted Hölder spaces.

We set $W_q^k(M;\rho) := W_q^{k,0}(M;\rho)$ and $BC^k(M;\rho) := BC^{k,0}(M;\rho)$. Also $W_q^k(\widehat{M}) := W_q^k\bigl((M,\widehat{g})\bigr)$, etc., and $L_q^\lambda(M;\rho) := W_q^{0,\lambda}(M;\rho)$.

Theorem 6.1. *If* $k \in \mathbb{N}$*, then*

$$W_q^k(M;\rho) \doteq W_q^k(\widehat{M}), \ 1 \le q < \infty,$$

and

$$BC^k(M;\rho) \doteq BC^k(\widehat{M}).$$

Proof. Let $0 \le j \le k$ and set $a_j^j := 1$. It follows from Remark 5.2(d) and (5.5) that

$$\begin{aligned} |\widehat{\nabla}^j u|_{\widehat{g}_0^j} &\le \sum_{i=0}^j |a_i^j \bullet \nabla^i u|_{\widehat{g}_0^j} = \sum_{i=0}^j |\rho^j a_i^j \bullet \nabla^i u|_{g_0^j} \\ &\le \sum_{i=0}^j |\rho^{j-i} a_i^j|_{g_0^{j-i}} \, |\rho^i \nabla^i u|_{g_0^i}. \end{aligned}$$

From this and (5.4) we get

$$|\widehat{\nabla}^j u|_{\widehat{g}_0^j} \le c(j) \sum_{i=0}^j |\rho^i \nabla^i u|_{g_0^i}.$$

Now we employ Remark 5.2(d) once more to deduce that

$$\bigl\| \, |\widehat{\nabla}^j u|_{\widehat{g}_0^j} \bigr\|_{L_p(\widehat{M})} \le c(j) \sum_{i=0}^j \bigl\| \rho^{i-m/p} \, |\nabla^i u|_{g_0^i} \bigr\|_{L_p(M)}, \ 1 \le p \le \infty.$$

By summing these inequalities from 0 to k and using (6.1), (6.2), we find that

$$\|u\|_{W_q^k(\widehat{M})} \le c(k) \, \|u\|_{W_q^k(M;\rho)}, \quad \|u\|_{BC^k(\widehat{M})} \le c(k) \, \|u\|_{BC^k(M;\rho)}.$$

Similar reasoning, based on (5.6), shows that the norm of $W_q^k(\widehat{M})$, resp. $BC^k(\widehat{M})$, is stronger than the one of $W_q^k(M;\rho)$, resp. $BC^k(M;\rho)$. The theorem is proved. □

Corollary 6.2. *Definition 5.1*(iii) *is equivalent to* $d(\log\rho) \in BC^\infty(T^*\widehat{M})$*.*

In order to deal with the case where $\lambda \ne 0$ we need the subsequent commutator estimate.

Lemma 6.3. *Let $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Then*

$$\sum_{i=0}^{k} |\widehat{\nabla}^i(\rho^\lambda u)|_{\widehat{g}_0^i} \sim \sum_{i=0}^{k} \rho^\lambda \, |\widehat{\nabla}^i u|_{\widehat{g}_0^i}.$$

Proof. (1) First we note that $\log \rho^\lambda = \lambda \log \rho$ and the preceding corollary imply that, setting $\delta := \rho^\lambda$,

$$a := d(\log \delta) \in BC^\infty(T^*\widehat{M}).$$

Furthermore, $d\delta = \delta a$. Hence

$$\widehat{\nabla}(\delta u) = \delta \widehat{\nabla} u + d\delta \otimes u = \delta(\widehat{\nabla} u + a \otimes u). \tag{6.3}$$

(2) We claim that

$$\widehat{\nabla}^k(\delta u) = \delta\Bigl(\widehat{\nabla}^k u + \sum_{i=0}^{k-1} a_i^k \otimes \widehat{\nabla}^i u\Bigr) = \delta \widehat{\nabla}^k u + \sum_{i=0}^{k-1} a_i^k \otimes \delta \widehat{\nabla}^i u, \ k \in \mathbb{N}, \tag{6.4}$$

where

$$a_i^k \in BC^\infty(T^0_{k-i}\widehat{M}), \ 0 \le i \le k. \tag{6.5}$$

This follows by induction, similarly as in step (2) of the proof of Lemma 5.4. It yields

$$|\widehat{\nabla}^k(\delta u)|_{\widehat{g}_0^k} \le c(k) \sum_{i=0}^{k} \delta \, |\widehat{\nabla}^i u|_{\widehat{g}_0^i}, \ k \in \dot{\mathbb{N}}.$$

(3) As in the proof of Lemma 5.4, we look at

$$\widehat{\nabla}^j(\delta u) = \delta \widehat{\nabla}^j u + \sum_{i=0}^{j} a_i^j \otimes \delta \widehat{\nabla}^i u, \ 0 \le j \le k,$$

as a lower triangular system of linear equations in the unknowns $\delta \widehat{\nabla}^i u$, $0 \le i \le k$. Solving it by forward substitution, we find $b_i^k \in BC^\infty(T^0_{k-i}\widehat{M}), \ 0 \le i \le k-1$, satisfying

$$\delta \widehat{\nabla}^k u = \widehat{\nabla}^k(\delta u) + \sum_{i=0}^{k-1} b_i^k \otimes \widehat{\nabla}^i(\delta u).$$

From this we get

$$|\delta \widehat{\nabla}^k u|_{\widehat{g}_0^k} \le c(k) \sum_{i=0}^{k} |\widehat{\nabla}^i(\delta u)|_{\widehat{g}_0^i}, \ k \in \mathbb{N}.$$

The assertion follows. □

We denote by $\rho^\lambda W_q^s(\widehat{M})$ the image space of $W_q^s(\widehat{M})$ under the map $u \mapsto \rho^\lambda u$. Thus the regular distribution $u \in L_{1,\mathrm{loc}}(\widehat{M})$ belongs to $\rho^\lambda W_q^s(\widehat{M})$ iff $\rho^{-\lambda} u \in W_q^s(\widehat{M})$. Similar definitions apply to the other spaces under consideration. It is clear that the spaces $\rho^\lambda W_q^s(\widehat{M})$, etc., are Banach spaces.

Theorem 6.4. *Suppose that $\lambda \in \mathbb{R}$. The map $u \mapsto \rho^\lambda u$ is an isomorphism from*

(i) $BC^s(\widehat{M})$ *onto* $BC^{s,\lambda}(M;\rho)$, $s \geq 0$.
(ii) $W_q^s(\widehat{M})$ *onto* $W_q^{s,\lambda}(M;\rho)$, $1 \leq q < \infty$, $s \geq 0$.
(iii) $H_q^s(\widehat{M})$ *onto* $H_q^{s,\lambda}(M;\rho)$, $1 < q < \infty$, $s \geq 0$.
(iv) $B_{p,r}^s(\widehat{M})$ *onto* $B_{p,r}^{s,\lambda}(M;\rho)$, $1 \leq p, r \leq \infty$, $s \geq 0$.

Proof. Lemma 6.3 implies

$$\sum_{j=0}^{k} \big\| \, |\widehat{\nabla}^j(\rho^{-\lambda} u)|_{\widehat{g}_0^j} \big\|_{L_p(\widehat{M})} \sim \sum_{j=0}^{k} \big\| \rho^{-\lambda} \, |\widehat{\nabla}^j u|_{\widehat{g}_0^j} \big\|_{L_p(\widehat{M})}$$

for $1 \leq p \leq \infty$. It is a consequence of Lemma 5.4 that the second sum is equivalent to

$$\sum_{j=0}^{k} \big\| \rho^{-\lambda} \, |\nabla^j u|_{\widehat{g}_0^j} \big\|_{L_p(\widehat{M})}.$$

We infer from Remark 5.2(d) that

$$\big\| \rho^{-\lambda} \, |\nabla^j u|_{\widehat{g}_0^j} \big\|_{L_p(\widehat{M})} = \big\| \rho^{-\lambda+j-m/p} \, |\nabla^i u|_{g_0^j} \big\|_{L_p(M)}.$$

These considerations show that

$$\|u\|_{\rho^\lambda W_q^k(\widehat{M})} \sim \|u\|_{W_q^{k,\lambda}(M;\rho)}, \ k \in \mathbb{N}, \ 1 \leq q < \infty,$$

and

$$\|\cdot\|_{\rho^\lambda BC^k(\widehat{M})} \sim \|\cdot\|_{BC^{k,\lambda}(M;\rho)}.$$

This yields assertions (i) and (ii) if $s = k$. The remaining claims now follow by interpolation. □

Corollary 6.5. *The weighted Sobolev–Slobodeckii, Bessel potential, and Besov spaces enjoy the same density, embedding, and interpolation properties as their classical (non-weighted) counterparts on $(\mathbb{R}^m, g_m)$, resp. $(\mathbb{H}^m, g_m)$.*

Remark 6.6. Suppose that $\lambda_1 \geq \lambda_0$. Then

$$W_q^{s,\lambda_1}(M) \hookrightarrow W_q^{s,\lambda_0}(M), \ s \geq 0, \ 1 \leq q < \infty.$$

Analogous embeddings apply to Bessel potential and Besov spaces.

Proof. If $s = k \in \mathbb{N}$, then $\rho^{-\lambda_0} = \rho^{-\lambda_1}\rho^{\lambda_1-\lambda_0} \leq \rho^{-\lambda_1}$ and (6.1) yield the assertion. The general case follows by interpolation, and the rest is now clear. □

Besides the interpolation results which are an outflow of the preceding corollary, there is also the following theorem for the case of different parameters. For simplicity, we consider only weighted Sobolev–Slobodeckii spaces. A corresponding statement applies to Bessel potential spaces (with $(\cdot,\cdot)_{\theta,q}$ replaced by $[\cdot,\cdot]_\theta$).

Theorem 6.7. *Suppose that* $s_0, s_1 \in \mathbb{R}_+$ *with* $s_0 \neq s_1$, $\theta \in (0,1)$, *and* λ_0, λ_1 *belong to* $\mathbb{R}$. *Set* $s_\theta := (1-\theta)s_0 + \theta s_1$ *and* $\lambda_\theta := (1-\theta)\lambda_0 + \theta\lambda_1$. *Then*

$$\big(W_q^{s_0,\lambda_0}(M;\rho), W_q^{s_1,\lambda_1}(M;\rho)\big)_{\theta,q} \doteq W_q^{s_\theta,\lambda_\theta}(M;\rho),$$

provided either $s_\theta \notin \mathbb{N}$ *or* $q=2$.

Proof. Let $\mathfrak{K}$ be a ur atlas for M. Set $\rho_\kappa := \kappa_*\rho(0) = \rho\big(\kappa^{-1}(0)\big)$. It is not difficult to see that $\kappa_*\rho \sim \rho_\kappa$ uniformly w.r.t. $\kappa \in \mathfrak{K}$. Using this 'discretization', one shows that $(\mathcal{R}, \mathcal{R}^c)$ is an r-c pair for $\big(\ell_q(\boldsymbol{\rho}^\lambda \boldsymbol{W}_q^s(\mathbb{M}), \rho^\lambda W_q^s(\widehat{M})\big)$, where

$$\boldsymbol{\rho}^\lambda \boldsymbol{W}_q^s(\mathbb{M}) := \prod_{\kappa\in\mathfrak{K}} \rho_\kappa^\lambda W_q^s(\mathbb{M}_\kappa).$$

Since $W_q^s = B_q^s$ if either $s \notin \mathbb{N}$ or $q=2$, we get the assertion by applying Corollary VI.2.3.3(i) and the isotropic version of Theorems VII.2.7.2(i), VII.2.7.4, and VII.2.8.3 in [8]. □

Remark 6.8. Let $\sigma, \tau \in \mathbb{N}$ and define

$$\|u\|_{W_q^{k,\lambda}(T_\tau^\sigma M;\rho)} := \sum_{j=0}^{k} \big\|\rho^{-\lambda+\tau-\sigma+i-m/q}\, |\nabla^i u|_{g_\sigma^{\tau+i}}\big\|_{L_q(T_\tau^\sigma M)}.$$

An analogous specification yields $\|\cdot\|_{BC^{k,\lambda}(T_\tau^\sigma M;\rho)}$. We introduce weighted Sobolev spaces, $W_q^{k,\lambda}(T_\tau^\sigma M;\rho)$, etc., by mimicking the definitions for the spaces $W_q^{k,\lambda}(M;\rho)$, etc. Then the foregoing theorems apply in this setting also. □

Of great significance in the theory of differential equations on Riemannian manifolds are the continuity properties of the covariant derivative in the function spaces under consideration. We content ourselves with exhibiting the Sobolev–Slobodeckii space case, making now use of the preceding remark.

Theorem 6.9. *Let* $\lambda \in \mathbb{R}$, $1 \le q < \infty$, $\sigma, \tau \in \mathbb{N}$, *and* $s \ge 0$. *Then*

$$\nabla \in \mathcal{L}\big(W_q^{s+1,\lambda}(T_\tau^\sigma M;\rho), W_q^{s,\lambda}(T_{\tau+1}^\sigma M;\rho)\big).$$

Proof. Let $k \in \mathbb{N}$ and $u \in W_q^{k+1,\lambda}(T_\tau^\sigma M;\rho)$. Then

$$\begin{aligned}\|\nabla u\|_{W_q^{k,\lambda}(T_{\tau+1}^\sigma M;\rho)} &= \sum_{i=0}^{k} \big\|\rho^{-\lambda+\tau+1-\sigma+i-m/q}\, |\nabla^i \nabla u|_{g_\sigma^{\tau+1+i}}\big\|_{L_q(M)} \\ &= \sum_{j=1}^{k+1} \big\|\rho^{-\lambda+\tau-\sigma-m/q}\, |\nabla^j u|_{g_\sigma^{\tau+j}}\big\|_{L_q(M)} \le \|u\|_{W_q^{k+1,\lambda}(T_\tau^\sigma M;\rho)}.\end{aligned}$$

This proves the assertion if $s=k$. If $s \notin \mathbb{N}$, then we conclude by interpolation. □

As for point-wise multiplications: the following theorem is an easy consequence of Theorems 4.6 and 6.4 and Remark 4.8(a).

Theorem 6.10. *Let* $\lambda_0, \lambda_1 \in \mathbb{R}$, $1 \le q < \infty$, *and* $\sigma, \tau, j \in \mathbb{N}$. *Then*

$$BC^{s_0,\lambda_0}(T_0^j M;\rho) \bullet W_q^{s_1,\lambda_1}(T_{\tau+j}^\sigma M;\rho) \hookrightarrow W_q^{s_1,\,\lambda_0+\lambda_1}(T_\tau^\sigma M;\rho)$$

if either $s_0 > s_1 \ge 0$ *or* $s_0 = s_1 \in \mathbb{N}$. *Also*

$$BC^{s_0,\lambda_0}(T_0^j M;\rho) \bullet BC^{s_1,\lambda_1}(T_{\tau+j}^\sigma M;\rho) \hookrightarrow BC^{s_1,\,\lambda_0+\lambda_1}(T_\tau^\sigma M;\rho)$$

with $0 \le s_1 \le s_0$. *If* $s > m/q$, *then*

$$W_q^{s,\lambda}(M;\rho) \bullet W_q^{s,\lambda}(M;\rho) \hookrightarrow W_q^{s,\,2\lambda}(M;\rho).$$

Thus $W_q^{s,\lambda}(M;\rho)$ *is a multiplication algebra for* $s > m/q$ *iff* $\lambda = 0$.

We refrain from formulating the weighted analog of the trace Theorem 4.7 and refer instead to [4] and [12]. It should be noted that the present definition (6.1) differs from the one we employed in our earlier papers by the factor $\rho^{-m/q}$. This change allows for simpler formulations of the point-wise multiplier theorems, for example.

7. Differential Operators

The importance of Theorem 6.4 is manifest by its corollary. It is also of fundamental importance in the study of differential equations on singular manifolds. Namely, it allows to carry over to the singular setting all existence and regularity theorems which can be derived in the less intricate frame of urR manifolds. Although this would lead too far afield, we explain the procedure in a simple setting and show just one consequence.

Suppose that $s \ge 0$, $\lambda, \mu \in \mathbb{R}$, and $\overline{s} \ge s$ with $\overline{s} > s$ if $s \notin \mathbb{N}$. Given

$$\boldsymbol{a} = (a_0, \dots, a_k) \in \prod_{j=0}^k BC^{\overline{s},\,\mu-\lambda}(T_0^j M;\rho),\ a_k \ne 0, \tag{7.1}$$

set

$$\mathcal{A} = \mathcal{A}(\boldsymbol{a}, \nabla) := \sum_{j=0}^k a_j \bullet \nabla^j,$$

where $(a_j \bullet \nabla^j)u := a_j \bullet (\nabla^j u)$. Then

$$\mathcal{A} \in \mathcal{L}\big(W_q^{s+k,\,\lambda}(M;\rho), W_q^{s,\mu}(M;\rho)\big) \tag{7.2}$$

by Theorems 6.9 and 6.10. Theorem 6.4 guarantees that

$$\mathsf{P}_s^\lambda := (u \mapsto \rho^\lambda u) \in \mathcal{L}\mathrm{is}\big(W_q^s(\widehat{M}), W_q^{s,\lambda}(M;\rho)\big).$$

Hence

$$\widehat{\mathcal{A}} := \mathsf{P}_s^{-\mu} \circ \mathcal{A} \circ \mathsf{P}_{s+k}^\lambda \in \mathcal{L}\big(W_q^{s+k}(\widehat{M}), W_q^s(\widehat{M})\big).$$

By means of (5.6) and (6.4) we derive that

$$\widehat{\mathcal{A}} = \mathcal{A}(\widehat{a}, \widehat{\nabla}) = \sum_{j=0}^k \widehat{a}_j \bullet \widehat{\nabla}$$

with
$$\widehat{\boldsymbol{a}}=(\widehat{a}_0,\dots,\widehat{a}_k)\in\prod_{j=0}^{k}BC^{\overline{s}}(T_0^j\widehat{M}).$$
Moreover, the map $\boldsymbol{a}\mapsto\widehat{\boldsymbol{a}}$ is an isomorphism. Given $f\in W_q^{s,\mu}(M;\rho)$,
$$u\in W_q^{s+k,\,\lambda}(M;\rho)\text{ satisfies }\mathcal{A}(\boldsymbol{a},\nabla)u=f$$
iff
$$\widehat{u}=\mathsf{P}_{s+k}^{-\lambda}u\in W_q^{s+k}(\widehat{M})\text{ complies with }\mathcal{A}(\widehat{a},\widehat{\nabla})\widehat{u}=\mathsf{P}_s^{-\mu}f.$$
The dependence of $\widehat{\boldsymbol{a}}$ on $\boldsymbol{a}$ is rather intricate. However, it is important that the leading coefficients a_k of $\mathcal{A}$ and $\widehat{a}_k$ of $\widehat{\mathcal{A}}$ enjoy the transparent relationship $\widehat{a}_k=\rho^{\lambda-\mu}a_k$.

Example 7.1. We consider the Laplace–Beltrami operator $\mathcal{A}:=\Delta=\operatorname{div}\operatorname{grad}$ on (M,g). We have to express it in terms of ∇. For this we use that the divergence of tensor fields is the linear map
$$\operatorname{div}\colon C^\infty(T_\tau^\sigma M)\to C^\infty(T_\tau^{\sigma-1}M),\ \sigma\in\dot{\mathbb{N}},\ \tau\in\mathbb{N},$$
defined by
$$\operatorname{div}a:=\mathsf{C}_{\tau+1}^{\sigma}(\nabla a),\ a\in C^\infty(T_\tau^\sigma M).$$
Hence
$$\begin{aligned}\Delta u=\mathsf{C}_1^1\big(\nabla(g^\sharp du)\big)&=\mathsf{C}_1^1\big(\nabla\mathsf{C}_1^2(g^*\otimes du)\big)\\&=\mathsf{C}_1^1\mathsf{C}_1^2(g^*\otimes\nabla du)=g^*\bullet\nabla^2u,\end{aligned}$$
since ∇ commutes with contractions and $\nabla g^*=0$. Observe that $|g^*|_{g_2^0}=m$. This and Remark 6.8 imply that $a_2\in BC^{\infty,\,-2}(T_0^2M;\rho)$. Thus $\mu=\lambda-2$ and
$$\mathcal{A}\in\mathcal{L}\big(W_q^{s+2,\,\lambda}(M;\rho),W_q^{s,\,\lambda-2}(M;\rho)\big),\ s\geq 0.$$
Consequently,
$$\widehat{\mathcal{A}}=\mathsf{P}_s^{2-\lambda}\circ\mathcal{A}\circ\mathsf{P}_{s+2}^{\lambda}=\widehat{a}_2\bullet\widehat{\nabla}^2+\widehat{a}_1\bullet\widehat{\nabla}+\widehat{a}_0.$$
(An explicit computation shows that the lower order coefficients are not zero.) Using $\widehat{a}_2=\rho^{\lambda-\mu}a_2=\rho^2g^*=\widehat{g}^*$, we see that, setting $\widehat{\Delta}:=\Delta_{\widehat{g}}$,
$$\widehat{\mathcal{A}}=\widehat{\Delta}+\widehat{a}_1\bullet\widehat{\nabla}+\widehat{a}_0,\ \widehat{a}_i\in BC^\infty(T_0^i\widehat{M}).\qquad\square$$

Mapping properties for elliptic and parabolic differential operators on urR manifolds have been investigated in [6], [7]. Based on those results and the preceding example, we can easily derive corresponding statements in the weighted space setting. We content ourselves by giving just one of the many possibilities.

Assume that $\partial M=\emptyset$ and fix a nontrivial compact subinterval J of $\mathbb{R}_+$ containing 0. We look at the initial value problem for the reaction-diffusion equation on M:
$$\partial_t u-\operatorname{div}(a\bullet\operatorname{grad}u)=f\text{ on }M\times J,\ u|_{t=0}=u_0\text{ on }M.\tag{7.3}$$
Choose $\lambda\in\mathbb{R}$ and $1<q<\infty$ and assume that
$$(f,u_0)\in L_q\big(J,L_q^{\lambda-2}(M;\rho)\big)\times W_q^{2-2/q,\,\lambda-2/q}(M;\rho).\tag{7.4}$$

Furthermore,

$$a \in BC^1(T_1^1 M) \tag{7.5}$$

and there exists a constant $\underline{\varepsilon} > 0$ such that

$$(\eta|a\eta)_g \geq \underline{\varepsilon}\,|\eta|_g^2,\ \eta \in \Gamma(TM),$$

or, equivalently,

$$(\xi|a\xi)_{g^*} \geq \underline{\varepsilon}\,|\xi|_{g^*}^2,\ \xi \in \Gamma(T^*M). \tag{7.6}$$

This means that

$$\mathcal{A} := -\operatorname{div}(a \bullet \operatorname{grad}) \tag{7.7}$$

is a uniformly (strongly) elliptic differential operator on $C^2(M)$.

Theorem 7.2. *Problem 7.3 has for each (f, u_0) satisfying (7.4) a unique solution*

$$u = u(f, u_0) \in L_q\big(J, W_q^{2,\lambda}(M;\rho)\big) \cap W_q^1\big(J, L_q^{\lambda-2}(M;\rho)\big). \tag{7.8}$$

The map $(f, u_0) \mapsto u(f, u_0)$ is linear and continuous.

Proof. (1) Using local coordinates, for example, it is verified that

$$-\mathcal{A} = (g^\sharp a) \bullet \nabla^2 + g^\sharp \operatorname{div}(a) \bullet \nabla = a_2 \bullet \nabla^2 + a_1 \bullet \nabla,$$

where $g^\sharp a = \mathsf{C}_1^2(g^* \otimes a) \in C^1(T_0^2 M)$. We find

$$|a_2|_{g_2^0} = |g^\sharp a|_{g_2^0} \leq |g^*|_{g_2^0}\,|a|_{g_1^1} = m\,|a|_{g_1^1} \in BC(M).$$

This implies

$$a_2 \in BC^{0,-2}(T_0^2 M;\rho).$$

Similarly, by (7.5),

$$|a_1|_g = |g^\sharp \operatorname{div}(a)|_g \leq m\,|\operatorname{div} a|_{g^*} \leq m\,|\nabla a|_{g_1^2} \in BC(M),$$

which yields

$$\rho^2(\rho^{-1}\,|a_1|_g) = \rho\,|a_1|_g \leq |a_1|_g \in BC(M).$$

Hence

$$a_1 \in BC^{0,-2}(T_0^1 M;\rho).$$

As in Example 7.1, the leading coefficient of $-\widehat{\mathcal{A}} = -\mathsf{P}_0^{2-\lambda}\mathcal{A}\mathsf{P}_2^\lambda$ reads

$$\widehat{a}_2 = \rho^2 a_2 = \rho^2 g^\sharp a = \widehat{g}^\sharp a \in BC(T_0^2 \widehat{M}).$$

It follows from (7.6) that

$$\begin{aligned}\widehat{a}_2 \bullet (\xi \otimes \xi) &= \langle \xi, \widehat{a}_2 \xi\rangle = \rho^2 \langle \xi, (g^\sharp a)\xi\rangle \\ &= \rho^2(\xi|a\xi)_{g^*} \geq \underline{\varepsilon}\rho^2\,|\xi|_{g^*}^2 = \underline{\varepsilon}\,|\xi|_{\widehat{g}^*}^2\end{aligned}$$

for $\xi \in \Gamma(T^*M)$. Hence $\widehat{\mathcal{A}}$ is a uniformly elliptic operator on $C^2(\widehat{M})$.

(2) For abbreviation,

$$\widehat{W}_q^s := W_q^s(\widehat{M}), \quad W_q^{s,\lambda} := W^{s,\lambda}(M;\rho).$$

By Corollary 6.5 and Remark 6.6,

$$W_q^{2,\lambda} \overset{d}{\hookrightarrow} L_q^{\lambda-2}. \tag{7.9}$$

Theorems 4.4 and 6.7 imply

$$(\widehat{L}_q, \widehat{W}_q^2)_{1-1/q,\,q} \doteq \widehat{W}_q^{2-2/q}, \quad (L_q^{\lambda-2}, W_q^{2,\lambda})_{1-1/q,\,q} \doteq W_q^{2-2/q,\,\lambda-2/q}, \tag{7.10}$$

respectively. Since, by Theorem 6.4,

$$\mathsf{P}_0^{\lambda-2} \in \mathcal{L}\mathrm{is}(\widehat{L}_q, L_q^{\lambda-2}), \quad \mathsf{P}_2^{\lambda} \in \mathcal{L}\mathrm{is}(\widehat{W}_q^2, W_q^{2,\lambda}), \tag{7.11}$$

we obtain from (7.10) by interpolation, see Theorem 6.7, that

$$\begin{aligned}&\mathsf{P}_{2-2/q}^{\lambda-2/q} \text{ is an isomorphism from } \widehat{W}_q^{2-2/q} \doteq (\widehat{L}_q, \widehat{W}_q^2)_{1-1/q,\,q}\\ &\text{onto } (L_q^{\lambda-2}, W_q^{2,\lambda})_{1-1/q,\,q} \doteq W_q^{2-2/q,\,\lambda-2/q}.\end{aligned} \tag{7.12}$$

Hence assumption (7.4) yields

$$\widehat{u}_0 := \mathsf{P}_{2-2/q}^{2/q-\lambda} u_0 \in (\widehat{L}_q, \widehat{W}_q^2)_{1-1/q,\,q}. \tag{7.13}$$

(3) The point-wise extension of $\mathsf{P}_s^{\lambda} \in \mathcal{L}\mathrm{is}(\widehat{W}_q^s, W_q^{s,\lambda})$ over J is again denoted by P_s^{λ}. Then we get from (7.11) that

$$\mathsf{P}_0^{\lambda-2} \in \mathcal{L}\mathrm{is}\big(L_q(J, \widehat{L}_q), L_q(J, L_q^{\lambda-2})\big)$$

and

$$\mathsf{P}_2^{\lambda} \in \mathcal{L}\mathrm{is}\big(L_q(J, \widehat{W}_q^2), L_q(J, W_q^{2,\lambda})\big). \tag{7.14}$$

Thus (7.8) implies

$$\widehat{f} := \mathsf{P}_0^{2-\lambda} f \in L_q(J, \widehat{L}_q). \tag{7.15}$$

Step (1) guarantees that $\partial_t + \widehat{\mathcal{A}}$ is a uniformly parabolic differential operator on $C^2(\widehat{M})$. Thus (7.13), (7.15), and Theorem 1.23 in [7] guarantee that the initial value problem

$$\partial_t v + \widehat{\mathcal{A}} v = \widehat{f} \text{ on } M \times J, \quad v|_{t=0} = \widehat{u}_0 \text{ on } M \tag{7.16}$$

has a unique solution

$$\widehat{u} = \widehat{u}(\widehat{f}, \widehat{u}_0) \in L_q(J, \widehat{W}_q^2) \cap W_q^1(J, \widehat{L}_q) \tag{7.17}$$

and that $(\widehat{f}, \widehat{u}_0) \mapsto \widehat{u}$ is linear and continuous. Assertion (7.17) is equivalent to

$$(\widehat{u}, \partial_t \widehat{u}) \in L_q(J, \widehat{W}_q^2) \times L_q(J, \widehat{L}_q). \tag{7.18}$$

(4) We set

$$u := \mathsf{P}_2^{\lambda} \widehat{u}. \tag{7.19}$$

Then, by (7.9), (7.14), and (7.18),

$$u \in L_q(J, W_q^{2,\lambda}) \hookrightarrow L_q(J, L_q^{\lambda-2}). \tag{7.20}$$

Moreover, see (7.15),

$$(\partial_t + \widehat{\mathcal{A}})\widehat{u} = (\partial_t + \mathsf{P}_0^{2-\lambda} \mathcal{A} \mathsf{P}_2^{\lambda})\widehat{u} = \mathsf{P}_0^{2-\lambda} f \in L_q(J, \widehat{L}_q)$$

and (7.19) imply

$$\partial_t(\mathsf{P}_2^{-\lambda} u) = \mathsf{P}_2^{-\lambda} \partial_t u = \mathsf{P}_0^{2-\lambda}(-\mathcal{A} u + f) \in L_q(J, \widehat{L}_q). \tag{7.21}$$

Note that $0 < \rho \le 1$ yields

$$\begin{aligned}\|\partial_t u\|_{L_q(J,L_q^{\lambda-2})} &= \|\rho^{2-\lambda-m/q}\partial_t u\|_{L_q(J,L_q)}\\ &\le \|\rho^{-\lambda-m/q}\partial_t u\|_{L_q(J,L_q)} = \|\rho^{-\lambda}\partial_t u\|_{L_q(J,\widehat{L}_q)}.\end{aligned}$$

From this and (7.21) it follows that

$$\begin{aligned}\|\partial_t u\|_{L_q(J,L_q^{\lambda-2})} &\le \|\mathsf{P}_0^{2-\lambda}(-\mathcal{A}u+f)\|_{L_q(J,\widehat{L}_q)}\\ &\le c(\|\mathcal{A}u\|_{L_q(J,L_q^{\lambda-2})} + \|f\|_{L_q(J,L_q^{\lambda-2})}\\ &\le c\big(\|u\|_{L_q(J,W_q^{2,\lambda})} + \|f\|_{L_q(J,L_q^{\lambda-2})}\big),\end{aligned} \tag{7.22}$$

where we also used (7.2) and step (1). Consequently, we get from (7.20) that

$$u \in L_q(J,W_q^{2,\lambda}) \cap W_q^1(J,L_q^{\lambda-2}).$$

(5) We know from [2, Theorem III.4.10.2] that

$$L_q(J,\widehat{W}_q^2) \cap W_q^1(J,\widehat{L}_q) \hookrightarrow BUC\big(J,(\widehat{L}_q,\widehat{W}_q^2)_{1-1/q,\,q}\big) =: \widehat{\mathbb{E}}$$

and

$$L_q(J,\widehat{W}_q^{2,\lambda}) \cap W_q^1(J,L_q^{\lambda-2}) \hookrightarrow BUC\big(J,(L_q^{\lambda-2},W_q^{2,\lambda})_{1-1/q,\,q}\big) =: \widehat{\mathbb{E}}^\lambda.$$

Hence, setting $\gamma_0\widehat{u} := \widehat{u}|_{t=0}$,

$$\gamma_0\widehat{u} = \lim_{t\to 0}\widehat{u}(t) \text{ in } \widehat{\mathbb{E}}.$$

We deduce from (7.12) that

$$\mathsf{P}_{2-2/q}^{\lambda-2/q}\gamma_0\widehat{u} = \lim_{t\to 0}\mathsf{P}_{2-2/q}^{\lambda-2/q}\widehat{u}(t) = \lim_{t\to 0}u(t) = \gamma_0 u \text{ in } \mathbb{E}^\lambda.$$

Consequently,

$$\gamma_0 u = \mathsf{P}_{2-2/q}^{\lambda-2/q}\gamma_0\widehat{u} = u_0,$$

due to (7.13).

Since the last part of the assertion follows from (7.19), (7.22), and the linearity and continuity of $(\widehat{f},\widehat{u}_0) \mapsto \widehat{u}$, the theorem is proved. □

Let E_0 and E_1 be Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$. Then $\mathcal{H}(E_1,E_0)$ denotes the set of all $A \in \mathcal{L}(E_1,E_0)$ such that $-A$, considered as a linear operator in E_0 with domain E_1, is the infinitesimal generator of a strongly continuous analytic semigroup $\{\, e^{-tA}\ ;\ t \ge 0\,\}$ on E_0, that is, in $\mathcal{L}(E_0)$.

Theorem 7.3. $\mathcal{A} \in \mathcal{H}\big(W_q^{2,\lambda}(M;\rho), L_q^{\lambda-2}(M;\rho)\big)$.

Proof. Theorem 7.2 and [2, Remark III.4.10.9(b)]. □

Corollary 7.4. *There exist* $\omega > 0$ *such that*

$$\lambda + \mathcal{A} \in \mathcal{L}\mathrm{is}\big(W_q^{2,\lambda}(M;\rho), L_q^{\lambda-2}(M;\rho)\big),\ \ \lambda \ge \omega.$$

Proof. [2, Theorem I.2.2]. □

Remark 7.5. Theorem 7.2 is a maximal Sobolev space regularity statement. Such results form the basis for proving existence results for quasilinear parabolic equations on singular manifolds. In this connection the minimal regularity assumptions (7.1) are crucial.

Theorem 1.23 in [7] also contains a maximal Hölder space regularity theorem on urR manifolds. Y. Shao and G. Simonett [59] have implemented the Da Prato–Grisvard theorem [27] on continuous maximal regularity on urR manifolds and applied it to the Yamabe flow (also see [46], [47]). Clearly, building on these results, similarly as we proved Theorem 7.2, we can establish maximal Hölder space and continuous regularity theorems in weighted spaces. □

It should be observed that the operator (7.7) is non-degenerate. Linear parabolic problems with degenerate coefficients have been investigated in [6], [9], [11], and nonlinear ones by Y. Shao [56], [57], [58] and in [10].

The fact that the weighted Sobolev spaces on (M, g, ρ) can be expressed in terms of Sobolev spaces on $(M, \widehat{g})$ has also been noted in [14] and in M. Kohr and V. Nistor [43]. Based on this observation, the authors apply earlier results (e.g., [15], [36]) to derive regularity and isomorphism theorems for elliptic boundary value problems.

8. Model Cusps

As mentioned in the introduction, it remains to exhibit concrete classes of singular manifolds. This is done in this and the following two sections.

We begin with very simple but important one-dimensional urR manifolds. Hereafter,

- $I := (0, 1]$ and R is a cusp characteristic,

that is,

$$\begin{aligned} &\text{(i)} \quad R \in C^\infty\big(I, (0,1]\big),\ \textstyle\int_I dt/R(t) = \infty. \\ &\text{(ii)} \quad \|R^{j-1}\partial^j R\|_{BC(I)} \le c(j),\ j \in \dot{\mathbb{N}}. \end{aligned} \tag{8.1}$$

Examples 8.1. (a) (Power characteristics) $\mathsf{R}_\alpha(t) := t^\alpha,\ \ \alpha \ge 1.$

(b) (Exponential characteristics) $\exp_{[\alpha,\beta]}(t) := e^{\alpha(1-t^{-\beta})}$ for $\alpha, \beta > 0$. □

Proposition 8.2. *$(I,\ dr^2/R^2)$ is a urR manifold.*

Proof. We set $\rho(s) := \int_s^1 dt/R(t)$ for $s \in I$. Then $\rho \in \mathrm{Diff}(I, \mathbb{R}_+)$ and

$$(\rho^* ds)(r) = d\rho(r) = -dr/R(r).$$

Thus $\rho^*(\mathbb{R}_+, ds^2) = (I,\ dr^2/R^2)$. This shows that ρ is an isometric diffeomorphism from $(I,\ dr^2/R^2)$ onto $(\mathbb{R}_+, ds^2)$. Examples 3.4(a) and (d) yield the claim. □

Let $1 \le m \le \mathfrak{m}$. Assume that either

$$\begin{aligned} &B \text{ is a compact } (m-1)\text{-dimensional submanifold}\\ &\text{of the unit sphere } \mathbb{S}^{\mathfrak{m}-1} \text{ in } \mathbb{R}^{\mathfrak{m}} \text{ and}\\ &C(B) := \{\, tb \ ; \ t \in I,\ b \in B \,\} \subset \mathbb{R}^{\mathfrak{m}}, \end{aligned} \tag{8.2}$$

or

$$\begin{aligned} &B \text{ is a compact } (m-1)\text{-dimensional submanifold of } \mathbb{R}^{\mathfrak{m}-1} \text{ and}\\ &K(R,B) := \big\{\, \big(t, R(t)b\big) \ ; \ t \in I,\ b \in B \,\big\} \subset \mathbb{R} \times \mathbb{R}^{\mathfrak{m}-1} = \mathbb{R}^{\mathfrak{m}}. \end{aligned} \tag{8.3}$$

Here tb is identified with $t\iota_B(b)$, where ι_B denotes the inclusion map $B \hookrightarrow \mathbb{R}^{\mathfrak{m}}$ if (8.2) applies, resp. $B \hookrightarrow \mathbb{R}^{\mathfrak{m}-1}$ otherwise. For the sake of a uniform presentation we write

$Z = Z(R,B)$ for either $C(B)$ or $K(R,B)$,
where it is understood that $R = \mathsf{R}_1$ in the first case.

It is said to be a *smooth model cusp* (*cone* if $Z = C$) *in* $\mathbb{R}^{\mathfrak{m}}$. 'Smooth' expresses the fact that the *base* B is a smooth manifold.

Remark 8.3. It is not assumed that B be connected. For examle, if $m = 1$ and $\mathfrak{m} = 3$, then B is a 0-dimensional submanifold of $\mathbb{S}^2$ if $Z = C$, resp. of $\mathbb{R}^2$ otherwise. Hence $B = \{b_1, \dots, b_k\}$ for some $k \in \dot{\mathbb{N}}$. If $Z = C$, then Z consists of k straight lines of length 1 in $\mathbb{R}^3$ emanating from the origin, but not containing it. Assume that $Z = K$. Then Z is a pair-wise disjoint union of k smooth curves of finite length in $\mathbb{R}^3$, which originate from $0 \in \mathbb{R}^3$ also. Given any two $a, b \in B$, write E for the plane containing 0, a, b. It encompasses the two curves with the endpoints a and b. If $\alpha = 1$, then they approach the origin transversally, that is, they form an angle in E. Otherwise, they determine a cusp in E, that is, Z represents a 'bouquet of flowers'. □

The map

$$f_Z \colon I \times B \to Z \quad (t,b) \mapsto \begin{cases} tb, & \text{if } Z = C,\\ \big(t, R(t)b\big), & \text{otherwise,} \end{cases}$$

is a diffeomorphism, the *stretching diffeomorphism*. Note that $I \times B$ is a manifold with corners if $\partial B \ne \emptyset$. Hence $Z = f_Z(I \times B)$ is such a manifold also. It is easy to see that everything established from Section 3 onwards extends naturally to such corner manifolds. (In this case $Q_\kappa^m := [0,1)^2 \times (-1,1)^{m-2}$ has to be added to (3.1) if U_κ is a neighborhood of a corner point.)

On B we introduce the pull-back metric $g_B := \iota_B^* g_\ell$, where $\ell = \mathfrak{m}$ if $Z = C$, and $\ell = \mathfrak{m} - 1$ otherwise. We endow Z with the metric

$$g_Z := f_{Z\,*}(dt^2 + R^2 g_B). \tag{8.4}$$

It is equivalent to the metric induced by the embedding $\iota_Z \colon Z \hookrightarrow \mathbb{R}^{\mathfrak{m}}$:

Lemma 8.4. $g_Z \sim \iota_Z^* g_{\mathfrak{m}}$.

Proof. We identify f_Z with $\iota_Z \circ f_Z$.

(1) Suppose that $Z = K$. Let $\mathfrak{K}$ be a ur atlas for B and put

$$h_\kappa := \iota_B \circ \kappa^{-1} \in C^\infty(Q_\kappa^{m-1}, \mathbb{R}^{\mathfrak{m}-1}),\ \ \kappa \in \mathfrak{K}.$$

Then

$$f_\kappa := \big((t,y) \mapsto (t, R(t)h_\kappa(y))\big) \in C^\infty(I \times Q_\kappa^{m-1}, \mathbb{R}^{\mathfrak{m}})$$

is a local parametrization of Z. From now on, the indices i and α run from 2 to $\mathfrak{m}$ and from 2 to m, respectively. Then

$$df_\kappa^1 = dt, \quad df_\kappa^i = \dot{R} h_\kappa^i dt + R\, \partial_\alpha h_\kappa^i dy^\alpha.$$

Hence, writing $|\cdot| = |\cdot|_{\mathfrak{m}}$,

$$\begin{aligned} f_\kappa^* g_{\mathfrak{m}} &= dt^2 + \sum_i |\dot{R} h_\kappa^i dt + R\, \partial_\alpha h_\kappa^i dy^\alpha|^2 \\ &= (1 + \dot{R}^2 |h_\kappa|^2) dt^2 + 2R\dot{R} dt \sum_i h_\kappa^i\, \partial_\alpha h_\kappa^i dy^\alpha + R^2 \sum_i |\partial_\alpha h_\kappa^i dy^\alpha|^2. \end{aligned}$$

Since B is compact, we can fix $\delta \ge 1$ such that $|h_\kappa| \le \delta$ for $\kappa \in \mathfrak{K}$. The Cauchy–Schwarz inequality yields

$$\Big| 2R\dot{R} dt \sum_i h_\kappa^i\, \partial_\alpha h^i dy^\alpha \Big| \le \varepsilon^{-1} \dot{R}^2 |h_\kappa|^2 dt^2 + \varepsilon R^2 \sum_i |\partial_\alpha h_\kappa^i dy^\alpha|^2 \ .$$

This implies that $f_\kappa^* g_{\mathfrak{m}}$ can be estimated from below by

$$\big(1 + (1 - \varepsilon^{-1}) \dot{R}^2 |h_\kappa|^2\big) dt^2 + (1 - \varepsilon) R^2 \sum_i |\partial_\alpha h_\kappa^i dy^\alpha|^2.$$

We fix $\varepsilon \in (0,1)$ close to 1 such that $(\varepsilon^{-1} - 1)\, \|\dot{R}\|_\infty^2\, \delta^2 < 1$. This guarantees, due to $\kappa_* g_B = h_\kappa^* g_{\mathfrak{m}-1}$, that

$$f_\kappa^* g_{\mathfrak{m}} \ge \Big(dt^2 + R^2 \sum_i |\partial_\alpha h_\kappa^i dy^\alpha|^2 \Big) \Big/ c = (dt^2 + R^2 \kappa_* g_B)/c.$$

A similar argument with $\varepsilon = 1$ yields

$$f_\kappa^* g_{\mathfrak{m}} \le c(dt^2 + R^2 \kappa_* g)$$

κ-uniformly. The claim follows.

(2) Assume $Z = C$. Then $f_\kappa(t,y) = t h_\kappa(y)$ for $y \in Q_\kappa^{m-1}$, where now $h_\kappa(y) \in \mathbb{S}^{\mathfrak{m}-1} \hookrightarrow \mathbb{R}^{\mathfrak{m}}$. Hence $|h_\kappa| = 1$. Consequently, letting j run from 1 to $\mathfrak{m}$ and β from 1 to m,

$$2 \sum_j h_\kappa^j\, \partial_\beta h_\kappa^j = \sum_j \partial_\beta (h_\kappa^j)^2 = 0.$$

From this, $\partial_t f_\kappa = h_\kappa$, and $\partial_\beta f_\kappa = t\, \partial_\beta h_\kappa$ we get

$$f_\kappa^* g_{\mathfrak{m}} = dt^2 + t^2 \kappa_* g_B,\ \ \kappa \in \mathfrak{K}. \tag{8.5}$$

Thus the assertion applies in this case also. □

Note that, by (8.5),

$$g_C = \iota_C^* g_{\mathfrak{m}}. \tag{8.6}$$

We define $r_Z \in C^\infty\big(Z,(0,1]\big)$ by

$$r_Z(x) := \begin{cases} |x|_{\mathfrak{m}}, & \text{if } Z = C, \\ R(x^1), & \text{otherwise.} \end{cases} \tag{8.7}$$

Theorem 8.5. *Set $\widehat{g}_Z := g_Z/r_Z^2$. Then $(M,\widehat{g}_Z)$ is a urR manifold.*

Proof. Since $r_Z = f_{Z*}R = R\circ f_Z^{-1}$,

$$\widehat{g}_Z = (f_{Z*}R)^{-2} f_{Z*}(dt^2 + R^2 g_B) = f_{Z*}(R^{-2}dt^2 + g_B).$$

Thus the claim follows from Proposition 8.2 and Examples 3.4(b)–(d). □

Theorem 8.6. *r_Z is a singularity function for (Z,g_Z).*

Proof. (1) We claim that

$$\|R^k\partial^k(\log R)\|_{BC(I)} \le c(\alpha,k),\ \ k\in\dot{\mathbb{N}}. \tag{8.8}$$

We proceed by induction.

Set $a := \partial(\log R)$ so that $Ra = \partial R$. Hence $|R\partial(\log R)| \le c(0)$. Thus the assertion is proved if $k=0$.

Assume $k\in\dot{\mathbb{N}}$ and

$$|R^{i+1}\partial^i a| \le c(\alpha,i),\ \ 0\le i\le k-1. \tag{8.9}$$

Then

$$\begin{aligned} R^{k+1}\partial^k a &= R^k\partial^k(Ra) - \sum_{i=0}^{k-1}\binom{k}{i} R^k\partial^i R\,\partial^{k-i}a \\ &= R^k\partial^{k+1}R - \sum_{i=1}^{k-1}\binom{k}{i}(R^{i-1}\partial^i R)(R^{k-i+1}\partial^{k-i}a)\ . \end{aligned}$$

Using this and (8.1)(ii), we deduce that (8.9) holds also for $i=k$. This proves the claim.

(2) Set $g_0 := dt^2 + R^2 g_B$. Then

$$\nabla_{g_0}\omega = \partial_t\omega_1 \oplus \nabla_{g_B}\omega_2,\ \ \omega = \omega_1\oplus\omega_2 \in T^*I\oplus T^*B = T^*(I\times B).$$

Hence $\nabla_{g_0}^k\omega = \partial_t^k\omega_1$ if $\omega_2 = 0$. Consequently,

$$\begin{aligned} &\big\| r_Z^{k+1}\,|\nabla_{g_Z}^k d(\log r_Z)|_{(g_Z)_0^{k+1}}\,\big\|_{BC(Z)} \\ &\qquad = \big\| f_{Z*}\big(R^{k+1}\,|\partial_t^{k+1}(\log R)|\big)\,\big\|_{BC(Z)} \\ &\qquad = \|R^{k+1}\partial_t^{k+1}(\log R)\|_{BC(I)} \le c(\alpha,k) \end{aligned}$$

for $k\in\mathbb{N}$ by step (1). Due to Theorem 8.5, the assertion follows. □

9. Manifolds with Point Singularities

We denote, for $m \in \dot{\mathbb{N}}$, by $\mathbb{B}^m$ the open unit ball in $\mathbb{R}^m = (\mathbb{R}^m, g_m)$.

Let S be a 0-dimensional submanifold of $\mathbb{R}^{\mathfrak{m}}$. This means that each $p \in S$ has an open neighborhood $\mathcal{U}_p$ such that $\mathcal{U}_p \cap S = \{p\}$. The uniform regularity of $(\mathbb{R}^{\mathfrak{m}}, g_{\mathfrak{m}})$ and the separability of $\mathbb{R}^{\mathfrak{m}}$ imply that

$$\inf\{\, |p-q|_{\mathfrak{m}} \ ;\ p, q \in \mathcal{S},\ p \neq q \,\} > 0.$$

This shows that part (iv) of the subsequent definition is meaningful.

Definition 9.1. Let $1 \le m \le \mathfrak{m}$. Assume:

(i) (M, g) is an m-dimensional Riemannian submanifold of $(\mathbb{R}^{\mathfrak{m}}, g_{\mathfrak{m}})$.

(ii) $\mathcal{S} = \mathcal{S}(M) := \mathrm{cl}_{\mathbb{R}^{\mathfrak{m}}}(M) \setminus M$ is a 0-dimensional submanifold of $\mathbb{R}^{\mathfrak{m}}$, the *singularity set of* M.

(iii) $\mathcal{Z}$ is a finite set of m-dimensional model cusps $Z = Z(R, B)$ in $\mathbb{R}^{\mathfrak{m}}$, and $Z(r) := Z \cap r\mathbb{B}^{\mathfrak{m}}$ for $0 < r < 1$.

(iv) There exist $\varepsilon \in (0,1)$, for each $p \in \mathcal{S}$ a model cusp $Z_p \in \mathcal{Z}$, an open neighborhood $\widetilde{U}_p$ of $\overline{Z_p(\varepsilon)}$ in $\mathbb{R}^{\mathfrak{m}}$, and an injective immersion $\widetilde{\psi}_p$ from $\widetilde{U}_p$ into $\mathbb{R}^{\mathfrak{m}}$ satisfying $\widetilde{\psi}_p(0) = p$,

$$M(p,\varepsilon) := \widetilde{\psi}_p\big(Z_p(\varepsilon)\big) \subset M,$$

$M(p,\varepsilon) \cap M(q,\varepsilon) = \emptyset$ for $p, q \in M$ with $p \neq q$, and, setting $\psi_p := \widetilde{\psi}_p | Z_p$,

$$\|\psi_p\|_{k,\infty} \le c(k),\ p \in \mathcal{S},\ k \in \dot{\mathbb{N}}. \tag{9.1}$$

Then $M(p,\varepsilon)$ is called *cusp of* M *at* p (although $p \notin M$) *of type* Z_p, and

$$M(\mathcal{S},\varepsilon) := \bigcup_{p \in \mathcal{S}} M(p,\varepsilon)$$

is a *cuspidal neighborhood of* M *along* $\mathcal{S}$ *of type* $\mathcal{Z}$. Also, (M, g) is a *(singular) manifold with (smooth) point singularities of type* $\mathcal{Z}$ if it possesses a cuspidal neighborhood along $\mathcal{S}$ of type $\mathcal{Z}$.

Of course, $M(p,\varepsilon)$ is called *cone* at p if its model cusp is a cone. □

We introduce a *cuspidal chart* φ_p for M at p by

$$\varphi_p := \psi_p^{-1} | M(p,\varepsilon) \colon M(p,\varepsilon) \to Z_p(\varepsilon),$$

which is a diffeomorphism, and $\{\, \varphi_p \ ;\ p \in \mathcal{S} \,\}$ is a *cuspidal atlas* for $M(\mathcal{S},\varepsilon)$. Henceforth,

$$\begin{aligned} &M \text{ is a manifold with point singularities and} \\ &M(\mathcal{S},\varepsilon) \text{ is a cuspidal neighborhood of type } \mathcal{Z}. \end{aligned} \tag{9.2}$$

Remarks 9.2. (a) Let $p \in \mathcal{S}$ and $Z_p = Z(R, B) \in \mathcal{Z}$. Then $\partial Z_p(\varepsilon) \neq \emptyset$ iff $\partial B \neq \emptyset$, and $\partial Z_p(\varepsilon) = Z_p(R, \partial B) \cap \varepsilon\mathbb{B}^{\mathfrak{m}}$. Thus

$$\partial M(p,\varepsilon) = \varphi_p^{-1}\big(\partial Z_p(\varepsilon)\big).$$

This shows that p is a singular point 'at the boundary of M'.

(b) Assume that p is a conical point of type $C(B)$. Then $\partial B = \emptyset$ iff $B = \mathbb{S}^{m-1} \hookrightarrow \mathbb{S}^{\mathfrak{m}-1}$. Since $C(\mathbb{S}^{m-1}) = \mathbb{B}^m \setminus \{0\}$, this means that '$M$ has a hole at p'.

(c) If p is a conical point of type $C(\mathbb{S}^{m-1} \cap \mathbb{H}^m)$, then '$p$ is a hole in the boundary ∂M'.

(d) A manifold with point singularities can have countably infinitely many cusps, but of finitely many types only. In this case M is unbounded. □

Let (N, h) be a Riemannian manifold and $A \subset N$. An atlas $\mathfrak{K}$ for N is *ur on* A if Definition 3.1 applies for $\kappa \in \mathfrak{K}_A := \{\, \nu \in \mathfrak{K} \ ;\ U_\nu \cap A \neq \emptyset \}$. Moreover, (N, h) is *ur on* A if Definition 3.3 holds with g replaced by h and $\mathfrak{K}$ with $\mathfrak{K}_A$. Note that $\mathfrak{K}_A$ is a finite set if A is compact and $\mathfrak{K}$ is ur on A.

We say that (M, g) is ur *off* $\mathcal{S}(M)$ if (M, g) is ur on

$$M(\mathcal{S}, r)^c := M \setminus M(\mathcal{S}, r)$$

for each $r \in (0, \varepsilon)$.

Examples 9.3. (a) A nonempty boundary ∂M is said to be *almost regularly embedded* if $m = \mathfrak{m}$ and $\partial M \cap M(\mathcal{S}, r)^c$ has for each $r \in (0, \varepsilon)$ a uniform geodesic collar. If this prevails, then (M, g) is ur off $\mathcal{S}(M)$.

Proof. This follows from the fact that there are ur atlases for $\mathbb{R}^m$ whose coordinate patches have arbitrarily small diameters. □

(b) Let M be bounded in $\mathbb{R}^{\mathfrak{m}}$. Then (M, g) is ur off $\mathcal{S}(M)$.

Proof. Since $M(\mathcal{S}, r)^c$ is compact, the localized version of Example 3.4(b) yields the assertion. □

(c) If $m = \mathfrak{m}$ and ∂M is relatively compact, then (M, g) is ur off $\mathcal{S}(M)$.

Proof. In this situation ∂M is almost regularly embedded. □

We fix $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon$ and an increasing function $\omega \in C^\infty[0, \varepsilon]$ satisfying $\omega(t) = 0$ if $t \le \varepsilon_0$, and $\omega(t) = 1$ for $t \ge \varepsilon_1$. Then

$$\omega_p := \varphi_p^* \omega \in C^\infty\bigl(M(p, \varepsilon), (0, 1]\bigr).$$

Hence

$$\rho_p := (1 - \omega_p) \varphi_p^* r_{Z_p} + \omega_p \in C^\infty\bigl(M(p, \varepsilon), (0, 1]\bigr),\ p \in \mathcal{S}. \tag{9.3}$$

The *cusp characteristic* ρ for M, defined by

$$\rho := \begin{cases} \rho_p \text{ on } M(p, \varepsilon), & p \in \mathcal{S}, \\ 1 \text{ on } M(\mathcal{S}, \varepsilon)^c, \end{cases} \tag{9.4}$$

belongs to $C^\infty\bigl(M, (0, 1]\bigr)$. We also introduce a Riemannian metric $\overline{g}$ on M by

$$\overline{g}_p := (1 - \omega_p) \varphi_p^* g_{Z_p} + \omega g,\ p \in \mathcal{S}, \tag{9.5}$$

and

$$\overline{g} := \begin{cases} \overline{g}_p \text{ on } M(p,\varepsilon), & p \in \mathcal{S}, \\ g \text{ on } M(\mathcal{S},\varepsilon)^c. \end{cases} \tag{9.6}$$

Theorem 9.4. *If (M, g) is ur off $\mathcal{S}(M)$ and $\widehat{g} := \overline{g}/\rho^2$, then $(M, \widehat{g})$ is ur.*

Proof. Set $M(p,r) := \varphi_p^{-1}\big(Z_p(r)\big)$, $0 < r < \varepsilon$, and $M(p,r] := \mathrm{cl}_M\big(M(p,r)\big)$.

(1) Theorem 8.5 guarantees that $(Z_p, \widehat{g}_{Z_p})$ is ur. Thus it is ur on

$$Z_p(\varepsilon_1] := Z_p \cap \varepsilon_1 \overline{\mathbb{B}}^{\mathfrak{m}}.$$

Example 3.4(d) yields that $\big(M(p,\varepsilon), \varphi_p^* \widehat{g}_{Z_p}\big)$ is ur on $M(p,\varepsilon_1]$. By the finiteness of $\mathcal{Z}$ and (9.1) this holds uniformly w.r.t. $p \in \mathcal{S}$.

(2) Note that

$$\rho_p = \varphi_p^* r_{Z_p},\ \overline{g}_p = \varphi_p^* g_{Z_p} \quad \text{on } Z_p(\varepsilon_0]. \tag{9.7}$$

Thus $\varphi_p^* \widehat{g}_{Z_p} = \widehat{g}$ on $M(p,\varepsilon_0]$.

(3) Since (M, g) is ur off $\mathcal{S}(M)$, (M, g) is ur on $M(\mathcal{S},\varepsilon_0)^c$. Note that $A_p := M(p,\varepsilon_1] \cap M(p,\varepsilon_0)^c$ is compact. Let $\overline{\mathfrak{K}}$ be an atlas for M which is ur on $M(\mathcal{S},\varepsilon_0)^c$. Its restriction to $M(p,\varepsilon)$ is an atlas which is ur on A_p. Hence $\overline{\mathfrak{K}}_p := \overline{\mathfrak{K}}_{A_p}$ is finite. Similarly, choose an atlas $\mathfrak{Z}_p$ for $Z_p(\varepsilon)$ which is ur on $Z_p(\varepsilon_1]$. Then $\varphi_p^* \mathfrak{Z}_p$ is an atlas for $M(p,\varepsilon)$ which is ur on $M(p,\varepsilon_1]$. Consequently, $\widehat{\mathfrak{K}}_p := (\varphi_p^* \mathfrak{Z}_p)_{A_p}$ is finite as well. Thus $\overline{\mathfrak{K}}_p \cup \widehat{\mathfrak{K}}_p$ is an atlas for the neighborhood

$$\bigcup_{\overline{\kappa} \in \overline{\mathfrak{K}}_p} U_{\overline{\kappa}} \cup \bigcup_{\widehat{\kappa} \in \widehat{\mathfrak{K}}_p} U_{\widehat{\kappa}}$$

of A_p which is finite and ur on A_p. From this we infer that $\big(M(p,\varepsilon), \overline{g}_p\big)$ is ur on A_p. Since $\min\{\, \rho_p(q)\ ;\ q \in A_p \,\} > 0$, it follows that $\widehat{g} \sim \overline{g}_p$ on A_p. Consequently, $(M, \widehat{g})$ is ur on A_p. The finiteness of $\mathcal{Z}$ and (9.1) guarantee that this holds uniformly w.r.t. $p \in \mathcal{S}$. From this and (9.3)–(9.6) the assertion follows. □

Remark 9.5. It is seen that a different choice of ε, the cut-off function ω, and of an equivalent cuspidal atlas leads to a ur metric for M which is equivalent to $\widehat{g}$. Thus this theorem means that we ‘uniformly regularize’ the singular manifold (M, g) by means of a ur metric $\widehat{g}$ which differs from g only arbitrarily close to the singularity set $\mathcal{S}(M)$. □

Let (M_i, g_i) be Riemannian manifolds and $f\colon (M_1, g_1) \to (M_2, g_2)$ be an isometric diffeomorphism. Set $\nabla_i := \nabla_{g_i}$ and define

$$f^*\nabla_2 \text{ by } (f^*\nabla_2)\omega = f^*\big(\nabla_2(f_*\omega)\big),\ \omega \in C^\infty(T^*M_1).$$

Then $f^*\nabla_2 = \nabla_1$ (see [12, Theorem X.2.3.2]). Hence

$$f^*\nabla_2^k = (f^*\nabla_2)^k = \nabla_1^k. \tag{9.8}$$

Theorem 9.6. *ρ is a singularity function for $(M, \overline{g})$.*

Proof. Due to Theorem 9.4, it remains to prove that

$$\|\rho^{k+1}\nabla_{\widehat{g}}^{k} d(\log\rho)\|_{BC(M)} \le c(k), \ k\in\mathbb{N}. \tag{9.9}$$

On $M(\mathcal{S},\varepsilon_0)^c$, $\underline{\rho}\le\rho\le 1$ for some $\underline{\rho}>0$. Therefore (9.9) is trivially true on $M(\mathcal{S},\varepsilon_0)^c$. Hence it remains to prove that (9.9) holds on $M(\mathcal{S},\varepsilon_0)$. Note that, by (9.7) and (9.8),

$$\varphi_{p*}\big(\rho^{k+1}\nabla_{\widehat{g}_p}^{k} d(\log\rho)\big) = r_{Z_p}^{k+1}\nabla_{g_{Z_p}}^{k} d(\log r_{Z_p}) \text{ on } Z_p(\varepsilon_0].$$

Thus the assertion follows from Theorem 8.6 and the finiteness of $\mathcal{Z}$. □

Manifolds with point singularities are the simplest class of singular manifolds. Larger families of uniformly regularizable singular manifolds are presented in [5] and in [12]. They comprise manifolds with cuspidal corners, cuspidal wedges, intruding cones, etc.

The starting point for a thorough analysis of singular manifolds and of differential operators thereon is the paper by V.A. Kondratiev [44]. Since then, there has appeared an inextricable flood of publications and it is impossible to do justice to the authors. We simply mention the prolific works of V.G. Maz′ya and B.-W. Schulze. Together with coworkers, the first author developed further Kondratiev's approach (see [45] for an early influential presentation). The main interest of the second author concerns algebras of pseudo-differential operators on manifolds with singularities. (Also the work by R.B. Melrose [48] should be mentioned). Schulze builds on the Mellin transform and cone differential operators of Fuchs-type (see [55], or the more recent book by V.E. Nazaikinskii, A.Yu. Savin, B.-W. Schulze, and B.Yu. Sternin [49] for accessible accounts). Both directions of research focus on singular function expansions which are not possible by our technique. E. Schrohe and coauthors develop a maximal regularity analysis of parabolic evolution equations on manifolds with conical singularities. They employ Schulze's Mellin–Sobolev spaces and implement the functional analytic Dore–Venni theorem (e.g., [2]). This requires the semigroup generator to have bounded imaginary powers, which is established for a class of cone differential operators containing the Laplace–Beltrami operator; see N. Roidos and E. Schrohe [51], E. Schrohe and J. Seiler [54], S. Coriasco, E. Schrohe, and J. Seiler [25], [26]. In contrast, Theorem 1.23 in [7], which is the fundament for Theorem 7.2, is based on a Fourier multiplier theorem on $\mathbb{R}^m$ and on the $(\mathcal{R},\mathcal{R}^c)$ localization technique, which apply with the same ease to parabolic equations of arbitrary order with values in spaces of sections of vector bundles.

10. Conical Singularities

Assume that $Z_p = C(B_p)$ with $p\in\mathcal{S}$. If $q\in M(p,\varepsilon)$, we write $d_p(q)=d_p^M(q)$ for the infimum of the lengths of all smooth curves $\gamma\colon[0,1]\to\mathbb{R}^{\mathfrak{m}}$ satisfying $\gamma(0)=p$, $\gamma(1)=q$, and $\gamma(0,1]\subset M(p,\varepsilon)$. Thus $d_p^M(q)$ is the Riemannian distance in $M(p,\varepsilon)$ from q to the 'point p at infinity' of $(M,\widehat{g})$.

Theorem 10.1. *Let (M,g) be an m-dimensional Riemannian submanifold of $(\mathbb{R}^{\mathfrak{m}}, g_{\mathfrak{m}})$ with conical singularities. Select a* distance function $\delta \in C^\infty(M)$ for $\mathcal{S}(M)$, *that is,*

$$\delta \sim d_p \text{ on } M(p,\varepsilon),\ p \in \mathcal{S}, \quad \delta = 1 \text{ on } M(\mathcal{S},\varepsilon)^c.$$

Then

$$u \mapsto \sum_{j=0}^{k} \left\| \delta^{-\lambda+j-m/q} \, |\nabla^j u|_{g_0^j} \right\|_{L_q(M)}, \ 1 \le q \le \infty,$$

is a norm for $W_q^{k,\lambda}(M;\rho)$ if $q<\infty$, resp. for $BC^{k,\lambda}(M;\rho)$ if $q=\infty$.

Proof. Let C_p be the model cone for $M(p,\varepsilon)$ and assume that φ_p is a conical chart. We infer from (8.6) and (8.7) that $r_{C_p} = |\cdot|_{\mathfrak{m}}$. Hence $d_0^{C_p} = r_{C_p}$. Consequently, $\varphi_p^* r_{C_p} = \varphi_p^* d_0^{C_p} \sim d_p^M$. Now the claim follows from (9.3). □

In order to not overstretch the present paper, we consider smooth singularities only. However, manifolds with corners, edges, interfaces, cuts, etc. play an important role, in numerical analysis in particular. In such situations it is also possible—though much more technical—to carry out a uniform regularization along the lines of the proof of Theorem 9.4 (see [12]).

A different approach is due to C. Băcuţă, A.L. Mazzucato, V. Nistor, and L. Zikatanov [23]. These authors employ a 'desingularization' technique for curvilinear polyhedra based on the theory of Lie manifolds introduced by B. Ammann, R. Lauter, and V. Nistor [17] and B. Ammann, A.D. Ionescu, and V. Nistor [16]. It is essentially equivalent to our uniform regularization technique. This implies that the general theory of weighted spaces, exposed in Sections 6 and 7, applies to general bounded curvilinear polyhedra and singularity functions which are equivalent to the distance to the full singularity set $\mathcal{S}(M)$.

Lastly, we consider the important classical Euclidean setting. Precisely, we suppose that

$$\begin{aligned}&(M,g) \text{ is an } m\text{-dimensional Riemannian submanifold}\\ &\text{of } (\mathbb{R}^m, g_m) \text{ with conical singularities}\\ &\text{such that } \partial M \text{ is almost regularly embedded.}\end{aligned} \tag{10.1}$$

The last condition is satisfied (see Example 9.3(c)) if ∂M is relatively compact, thus, in particular, if M is bounded, which is the case most often looked at in the literature.

Let δ be a distance function for $\mathcal{S}(M)$, $k \in \mathbb{N}$, $1 \le q \le \infty$, and $a \in \mathbb{R}$. The *Kondratiev space* $K_{q,a}^k(M)$ consists of all $u \in L_{1,\mathrm{loc}}(M)$ whose distributional derivatives (in $\mathring{M}$) of order at most k are such that

$$\|u\|_{K_{q,a}^k} := \sum_{|\alpha| \le k} \|\delta^{|\alpha|-a} \partial^\alpha u\|_{L_q(M)} < \infty.$$

It is endowed with this norm.

Theorem 10.2. *Let* (10.1) *be satisfied. Then*

$$K_{q,a}^k(M) \doteq \begin{cases} W_q^{k,\,a-m/q}(M;\rho), & \text{if } q<\infty,\\ BC^{k,a}(M;\rho), & \text{otherwise.}\end{cases}$$

Proof. Now $g = g_m$. Thus $\nabla^j u = \partial^j u \in C(M, \mathcal{L}^j)$ for $u \in C^j(M)$, where $\mathcal{L}^j$ is the space of j-linear functions on $\mathbb{R}^m$. It is given the usual norm $|\cdot|_{\mathcal{L}_j}$. Set $\lambda := a - m/q$. Then

$$\begin{aligned}\sum_{j=0}^k \big\|\delta^{-\lambda+j-m/q}\, |\nabla^j u|_{g_0^j}\big\|_{L_q(M)} &\sim \sum_{j=0}^k \big\|\delta^{j-a}\, |\partial^j u|_{\mathcal{L}^j}\big\|_{L_q(M)}\\ &\sim \sum_{|\alpha|\le k} \|\delta^{|\alpha|-k}\partial^\alpha u\|_{L_q(M)}.\end{aligned}$$

Since the distributional, that is weak, derivatives (on $\mathring{M}$) coincide with the strong ones, and ∂M is an m-dimensional Lebesgue 0-set, well-known arguments yield that $K_{q,a}^k(M)$ is the completion of $(\mathcal{D}(M), \|\cdot\|_{K_{q,a}^k})$ in $L_{1,\mathrm{loc}}(M)$ if $q<\infty$, resp. of $(BC^\infty(M), \|\cdot\|_{K_{\infty,a}^k})$ in $C(M)$ otherwise. Now the assertion is clear. □

Kondratiev spaces on subdomains of Euclidean spaces have recently gained some attraction, see St. Dahlke, M. Hansen, C. Schneider, and W. Sickel [28], [29], St. Dahlke and C. Schneider [30], [31], for example. (The references in these papers to earlier works of various other authors should also be taken into account.) These publications contain proofs of embedding, interpolation, and point-wise multiplier theorems for a subclass of the curvilinear polyhedra considered by V. Nistor and coauthors. Some investigations of Kondratiev spaces of fractional order are found in [38].

Due to Theorem 10.2 and the remarks following Theorem 10.1, the full theory exposed in Section 6 applies to the clasical Kondratiev spaces on curvilinear polyhedra.

Appendix: Notations and Conventions

We employ standard notations some of which we recall below. As for manifolds, we refer to [32]. A detailed exposition is given in [12].

We use c to denote a generic constant ≥ 1, whose value may be different in different formulas but is always independent of the free variables in a given setting. Real vector spaces are considered throughout. The complex case can be covered by complexification.

Let S be a nonempty set. On the vector space $\Gamma(S) := \mathbb{R}^S$ of all real-valued functions on S, an equivalence relation $\sim$ is defined by $f_1 \sim f_2$ iff $f_1/c \le f_2 \le c f_1$. If S is a subset of some vector space, then $\dot{S} := S \setminus \{0\}$.

Given Banach spaces E, E_1, and E_2, $\mathcal{L}(E_1, E_2)$ is the Banach space of bounded linear maps from E_1 into E_2, and $\mathcal{L}\mathrm{is}(E_1, E_2)$ is the open subset of isomorphisms. The dual of E is written E', $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_E \colon E' \times E \to \mathbb{R}$ is the duality pairing, and $a' \in \mathcal{L}(E_2', E_1')$ is the dual of $a \in \mathcal{L}(E_1, E_2)$. As usual, $E_1 \hookrightarrow E_2$ denotes continuous injection, and $\overset{d}{\hookrightarrow}$ says that E_1 is also dense in E_2. Furthermore, $E_1 \doteq E_2$ iff $E_1 \hookrightarrow E_2$ and $E_2 \hookrightarrow E_1$, that is, E_1 and E_2 are equal except for equivalent norms.

Let M be an m-dimensional manifold with (a possibly empty) boundary. We work in the smooth category and assume that the underlying topological space is separable and metrizable. If V is a vector bundle over M, then $\Gamma(V) = \Gamma(M,V)$ is the $\Gamma(M)$-module of sections of V (*no* smoothness). Thus $\Gamma(M) = \Gamma(M \times \mathbb{R})$, where $M \times \mathbb{R}$ is the trivial bundle. If V is a metric vector bundle, then $C^\infty(V)$ is the $C^\infty(M)$ submodule of smooth sections.

As customary, TM and T^*M are the tangent and cotangent bundles of M. Then $T^\sigma_\tau M := TM^{\otimes\sigma} \otimes T^*M^{\otimes\tau}$ is, for $\sigma, \tau \in \mathbb{N}$, the (σ,τ)-tensor bundle of M, that is, the vector bundle of all tensors on M being contravariant of order σ and covariant of order τ. In particular, $T^1_0 M = TM$ and $T^0_1 M = T^*M$, as well as $T^0_0 M = M \times \mathbb{R}$.

For $\nu \in \mathbb{N}^\times$ we put $\mathbb{J}_\nu := \{1,\ldots,m\}^\nu$. Then, given local coordinates $\kappa = (x^1,\ldots,x^m)$ on an open subset U of M and setting

$$\frac{\partial}{\partial x^{(i)}} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\sigma}}, \quad dx^{(j)} := dx^{j_1} \otimes \cdots \otimes dx^{j_\tau}$$

for $(i) = (i_1,\ldots,i_\sigma) \in \mathbb{J}_\sigma$, $(j) \in \mathbb{J}_\tau$, the local representation of a (σ,τ)-tensor field $a \in \Gamma(T^\sigma_\tau M)$ with respect to these coordinates is given by

$$a = a^{(i)}_{(j)} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)}$$

with $a^{(i)}_{(j)} \in \Gamma(U)$. We use the summation convention for (multi-)indices labeling coordinates or bases. Thus such a repeated index, which appears once as a superscript and once as a subscript, implies summation over its whole range.

Suppose $1 \le s \le \sigma$ and $1 \le t \le \tau$. We write $(i) \in \mathbb{J}_\sigma$ in the form $(i_s; i_{\widehat{s}})$ where we set $(i_{\widehat{s}}) := (i_1,\ldots,\widehat{i_s},\ldots,i_\sigma) \in \mathbb{J}_{\sigma-1}$. Then we define the *contraction map*, C^s_t, *with respect to positions s and t* by

$$\mathsf{C}^s_t a := a^{(k;i_{\widehat{s}})}_{(k;j_{\widehat{t}})} \frac{\partial}{\partial x^{(i_{\widehat{s}})}} \otimes dx^{(j_{\widehat{t}})}, \ a \in \Gamma(T^\sigma_\tau M),$$

where k runs from 1 to m. Hence $\mathsf{C}^s_t \colon C^\infty(T^\sigma_\tau M) \to C^\infty(T^{\sigma-1}_{\tau-1} M)$ and this map is linear.

Let $0 \le \rho \le \sigma$ and

$$(a,b) \in \Gamma(T^\sigma_\tau M \oplus T^0_\rho M) \ .$$

We write $(i) \in \mathbb{J}_\sigma$ in the form $(\ell)(k)$ with $(k) \in \mathbb{J}_\rho$. Then

$$a = a^{(\ell)(k)}_{(j)} \frac{\partial}{\partial x^{(\ell)}} \otimes \frac{\partial}{\partial x^{(k)}} \otimes dx^{(j)}, \ b = b_{(k)} dx^{(k)}.$$

The *(complete) contraction multiplication* is defined by

$$\Gamma(T^\sigma_\tau M)\times\Gamma(T^0_\rho M)\to\Gamma(T^{\sigma-\rho}_\tau M),\qquad (a,b)\mapsto a\bullet b \tag{A.1}$$

with

$$a\bullet b:=a^{(\ell)(k)}_{(j)}b_{(k)}\frac{\partial}{\partial x^{(\ell)}}\otimes dx^{(j)} \tag{A.2}$$

(so that (k) runs through $\mathbb{J}_\rho$). We infer from (A.2) that (A.1) is a bilinear map

$$C^\infty(T^\sigma_\tau M)\times C^\infty(T^0_\rho M)\to C^\infty(T^{\sigma-\rho}_\tau M).$$

Let g be a Riemannian metric on TM. We write $g_\flat\colon TM\to T^*M$ for the (fiber-wise) Riesz isomorphism. Thus $\langle g_\flat X,Y\rangle=g(X,Y)$ for X and Y in $\Gamma(TM)$ where $\langle\cdot,\cdot\rangle\colon\Gamma(T^*M)\times\Gamma(TM)\to\Gamma(M)$ is the natural (fiber-wise) duality pairing. Hence $T^*_pM=(T_pM)'$ for $p\in M$. The inverse of $g_\flat$ is denoted by $g^\sharp$. Then g^*, the adjoint Riemannian metric on T^*M, is defined by $g^*(\alpha,\beta):=g(g^\sharp\alpha,g^\sharp\beta)$ for $\alpha,\beta\in\Gamma(T^*M)$. In local coordinates

$$g=g_{ij}\,dx^i\otimes dx^j,\quad g^*=g^{ij}\,\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j},$$

$[g^{ij}]$ being the inverse of the $(m\times m)$-matrix $[g_{ij}]$. We also employ the notation $(\cdot|\cdot)_g:=g$.

The metric g induces a vector bundle metric on $T^\sigma_\tau M$ which we denote by g^τ_σ. In local coordinates

$$g^\tau_\sigma(a,b)=g_{(i)(j)}g^{(k)(\ell)}a^{(i)}_{(k)}b^{(j)}_{(\ell)},\ \ a,b\in\Gamma(T^\sigma_\tau M),$$

where

$$g_{(i)(j)}:=g_{i_1j_1}\cdots g_{i_\sigma j_\sigma},\quad g^{(k)(\ell)}:=g^{k_1\ell_1}\cdots g^{k_\tau\ell_\tau}$$

for $(i),(j)\in\mathbb{J}_\sigma$ and $(k),(\ell)\in\mathbb{J}_\tau$. Note $g^0_1=g$ and $g^1_0=g^*$ and $g^0_0(a,b)=ab$ for $a,b\in\Gamma(M)$. Moreover,

$$|\cdot|_{g^\tau_\sigma}\colon\Gamma(T^\sigma_\tau M)\to(\mathbb{R}_+)^M,\quad a\mapsto\sqrt{g^\tau_\sigma(a,a)}$$

is the *vector bundle norm* on $T^\sigma_\tau M$ induced by g. It follows that the complete contraction is a fiber-wise continuous bilinear map.

The Euclidean metric on $\mathbb{R}^m$ is named $g_m=(dx^1)^2+\cdots+(dx^m)^2$, and $|\cdot|_m:=|\cdot|_{g_m}$ is the Euclidean norm.

The Levi–Civita connection on TM is denoted by $\nabla=\nabla_g$. We use the same symbol for its natural extension to a metric connection on $T^\sigma_\tau M$. Then the corresponding covariant derivative is the linear map

$$\nabla\colon C^\infty(T^\sigma_\tau M)\to C^\infty(T^\sigma_{\tau+1}M),\quad a\mapsto\nabla a,$$

defined by $\langle\nabla a,b\otimes X\rangle:=\langle\nabla_X a,b\rangle$ for $b\in C^\infty(T^\tau_\sigma M)$ and $X\in C^\infty(TM)$. It is a well-defined continuous linear map from $C^1(T^\sigma_\tau M)$ into $C(T^\sigma_{\tau+1}M)$, as follows from its local representation with the Christoffel symbols. For $k\in\mathbb{N}$ we define

$$\nabla^k\colon C^\infty(T^\sigma_\tau M)\to C^\infty(T^\sigma_{\tau+k}M),\quad a\mapsto\nabla^k a$$

by $\nabla^0 a := a$ if $\sigma = \tau = 0$, and $\nabla^{k+1} := \nabla \circ \nabla^k$, where $\nabla^1 = d$, the differential on $C^\infty(M)$. If $(M, g) = (\mathbb{R}^m, g_m)$, then $\nabla_{g_m} = \partial = (\partial_1, \ldots, \partial_m)$, the (Fréchet) derivative on $\mathbb{R}^m$.

Let M_1 and M_2 be m-dimensional manifolds and $f\colon M_1 \to M_2$ a diffeomorphism, in symbols: $f \in \mathrm{Diff}(M_1, M_2)$. The *push-forward* of functions, $f_*\colon C^\infty(M_1) \to C^\infty(M_2)$, is the linear bijection defined by $f_* u := u \circ f^{-1}$ for $u \in C^\infty(M_1)$. Its inverse, the *pull-back* f^*, is given by $f^* v = v \circ f$ for $v \in C^\infty(M_2)$. The push-forward (by f) of vector fields is the vector space isomorphism

$$f_*\colon C^\infty(TM_1) \to C^\infty(TM_2), \quad X \mapsto f_* X$$

specified by

$$(f_* X)(q) := (T_{f^{-1}(q)} f) X\big(f^{-1}(q)\big), \ q \in M_2.$$

Here $T_p f\colon T_p M_1 \to T_{f(p)} M_2$ is the tangent map of f at $p \in M_1$. The pull-back of vector fields (by f),

$$f^*\colon C^\infty(TM_2) \to C^\infty(TM_1), \ Y \mapsto f^* Y,$$

is given by

$$(f^* Y)(p) := (T_p f)^{-1} Y\big(f(p)\big), \ p \in M_1.$$

The vector space isomorphisms

$$f_*\colon C^\infty(T^* M_1) \to C^\infty(T^* M_2), \quad \alpha \mapsto f_* \alpha,$$

introduced by

$$(f_* \alpha)(q) := (T_q f^{-1})' \alpha\big(f^{-1}(q)\big), \ q \in M_2,$$

and

$$f^*\colon C^\infty(T^* M_2) \to C^\infty(T^* M_1), \ \beta \mapsto f^* \beta,$$

with

$$(f^* \beta)(p) := (T_p f)' \beta\big(f(p)\big), \ p \in M_1,$$

are the push-forward and the pull-back, respectively, of covector fields.

Now we define

$$f_*\colon C^\infty(T^\sigma_\tau M_1) \to C^\infty(T^\sigma_\tau M_2)$$

inductively by

$$f_*(a \otimes b) := f_* a \otimes f_* b, \ a \in C^\infty(T^{\sigma-i}_{\tau-j} M), \ b \in C^\infty(T^i_j M),$$

where $i, j \in \{0, 1\}$ with $\sigma \geq i$ and $\tau \geq j$. An analogous definition applies to

$$f^*\colon C^\infty(T^\sigma_\tau M_2) \to C^\infty(T^\sigma_\tau M_1).$$

Then f_* is a vector space isomorphism and $(f_*)^{-1} = f^*$. Furthermore, if $f_1 \in \mathrm{Diff}(M_1, M_2)$, then $(f_1 \circ f_2)_* = f_{1\,*} \circ f_{2\,*}$.

Suppose that (M_1, g_1) and (M_2, g_2) are Riemannian manifolds. Then

$$f\colon (M_1, g_1) \to (M_2, g_2)$$

is an *isometric diffeomorphism* if $f \in \mathrm{Diff}(M_1, M_2)$ and $f_* g_1 = g_2$.

We write dV_g for the Riemann–Lebesgue volume measure on (M,g). In local coordinates $\kappa=(x^1,\dots,x^m)$ it is represented by $\kappa_* dV_g=(\kappa_*\sqrt{g})dx$, where $\sqrt{g}:=\big(\det[g_{ij}]\big)^{1/2}$ and dx is the Lebesgue measure on $\mathbb{R}^m$.

In Section 8 there occur product manifolds of the type $Z:=(0,1]\times B$, where B is an $(m-1)$-dimensional manifold. If $\partial B\neq\emptyset$, then $\{1\}\times\partial B$ is a 'corner' of Z, which is locally diffeomorphic to $[0,1)^2\times(-1,1)^{m-2}$. Everything said above has straightforward extensions to such manifolds with corners. Detailed investigations of manifolds with corners are given in [12].

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