Local and Global Strong Solutions to Continuous Coagulation-Fragmentation Equations with Diffusion

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Abstract

We consider the diffusive continuous coagulation-fragmentation equations with and without scattering and show that they admit unique strong solutions for a large class of initial values. If the latter values are small with respect to a suitable norm, we provide sufficient conditions for global-in-time existence in the absence of fragmentation.

Key words: coagulation, fragmentation, volume scattering, diffusion, semigroup theory

1 Introduction

The present paper contributes to the mathematical investigation of coagulation and fragmentation processes. Describing the mechanisms by which particles can merge to build larger particles or break up into smaller ones, these processes are met in various scientific and industrial disciplines such as physics, chemistry, biology, or oil and food industry (see [1], [10], [16] for a more detailed list of applications and for further references). In most situations, the particles are supposed to be fully identified by only one parameter indicating

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the particle's mass or volume. The continuous models differ from the discrete ones in that this parameter ranges in the set of all non-negative real numbers instead just in the positive integers. In this paper we focus on the former case although all of our results can be applied by analogy to the discrete one. In addition, we assume that the movement of the particles is driven by diffusion and thus neglect other effects such as external force fields. Besides the possibility that a particle can split spontaneously into smaller fragments, two particles can, as a result of their motion, collide with each other what then may lead to different outcomes. In the case of high-energy collisions, a shattering of the involved particles may occur, a process which is referred to as *collisional breakage* (cf. [8], [17]). On the other hand, the colliding particles may also stick together producing a particle of cumulative size. In most of the models studied in the physical and mathematical literature, the particles formed by coalescence may become arbitrarily large. Recently, a new and somewhat more consistent mechanism, called *(volume) scattering*, has been proposed in [11] for the particular case of two-phase liquids. There, the obvious fact is taken into account that particles cannot grow unrestrictedly. The idea is that, if the cumulative size of the colliding particles exceeds a certain maximal value, the particles coalesce merely virtually and decay instantaneously into particles all with size less than or equal to the maximal admissible size.

In this paper we treat the continuous diffusive coagulation-fragmentation equations with and without scattering simultaneously. More precisely, denoting by $u = u(y) = u(t, y, x)$ the particle size distribution function at time t and position x (where y is referring to the particle size), we consider the coupled reaction-diffusion equations

$$\begin{aligned} \partial_t u(y) - d(y)\Delta_x u(y) &= L[u](y) \ \textit{in } \Omega\ , \quad t > 0\ , \qquad y \in (0, y_0)\ , \\ \partial_\nu u(y) &= 0 \qquad \textit{on } \partial\Omega\ , \ t > 0\ , \qquad y \in (0, y_0)\ , \\ u(0, y, \cdot) &= u^0(y) \quad \textit{in } \Omega\ , \quad y \in (0, y_0)\ , \end{aligned} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, where $u^0 = u^0(y, x)$ is a given initial distribution, and where the reaction terms

$$L[u] := L_{\mathsf{b}}[u] + L_{\mathsf{c}}[u, u] + L_{\mathsf{s}}[u, u]$$

are defined by

$$\begin{aligned} L_{\mathsf{b}}[u](y) :=& \, L^1_{\mathsf{b}}[u](y) + L^2_{\mathsf{b}}[u](y) \\ :=& \int_y^{y_0} \gamma(y', y)\ u(y')\ \mathrm{d}y' - u(y) \ \int_0^y \frac{y'}{y}\ \gamma(y, y')\ \mathrm{d}y'\ , \end{aligned}$$

$$
\begin{aligned}
L_{\mathsf{c}}[u,v](y) :=&\, L_{\mathsf{c}}^1[u,v](y) + L_{\mathsf{c}}^2[u,v](y) - L_{\mathsf{c}}^3[u,v](y)\\
:=&\, \frac{1}{2}\int_0^y K(y',y-y')\ P(y',y-y')\ u(y-y')\ v(y')\ \mathrm{d}y'\\
&+\frac{1}{2}\int_y^{y_0}\int_0^{y'} K(y'',y'-y'')\ Q(y'',y'-y'')\\
&\qquad\qquad \times\ \beta_{\mathsf{c}}(y',y)\ u(y'')\ v(y'-y'')\ \mathrm{d}y''\mathrm{d}y'\\
&-u(y)\int_0^{y_0-y} K(y,y')\ \Big\{P(y,y')+Q(y,y')\Big\}\ v(y')\ \mathrm{d}y'\ ,\\
L_{\mathsf{s}}[u,v](y) :=&\, L_{\mathsf{s}}^1[u,v](y) - L_{\mathsf{s}}^2[u,v](y)\\
:=&\, \frac{1}{2}\int_{y_0}^{2y_0}\int_{y'-y_0}^{y_0} K(y'',y'-y'')\ \beta_{\mathsf{s}}(y',y)\ u(y'')\ v(y'-y'')\ \mathrm{d}y''\mathrm{d}y'\\
&-u(y)\int_{y_0-y}^{y_0} K(y,y')\ v(y')\ \mathrm{d}y'\ ,
\end{aligned}
$$

for $y \in (0,y_0)$. Here, the particles either have an upper bound $y_0 \in (0,\infty)$ (the case with scattering) or may become arbitrarily large if $y_0 = \infty$ (the case without scattering). In the latter case, the operator L_{s} is assumed to be identically zero. It is of importance to point out that we treat the two situations $y_0 < \infty$ and $y_0 = \infty$ simultaneously in the following.

The meaning of the operators $L_{\mathsf{b}}, L_{\mathsf{c}}$, and L_{s} are as follows:

- The linear operator $L_{\mathsf{b}}[u]$ accounts for the gain and loss of particles of size y due to multiple spontaneous breakage, where $\gamma(y,y') \geq 0$ denotes the rate at which a particle of size $y \in Y$ decays into a particle of size $y' \in (0,y)$.
- The possible events that may happen if two particles y and y' with cumulative size $y+y' < y_0$ collide, are reflected by the operator $L_{\mathsf{c}}[u,u]$. Colliding at the rate $K(y,y') \geq 0$, they may either merge with probability $P(y,y')$ or shatter into several particles according to the distribution $\beta_{\mathsf{c}}(y+y',y'')$ with probability $Q(y,y')$. The latter process is called collisional breakage. Clearly, consistency of the model then demands that

$$
0 \leq P(y,y') + Q(y,y') \leq 1\ , \quad y+y' < y_0\ . \tag{2}
$$

- Finally, if $y_0 < \infty$, the scattering operator $L_{\mathsf{s}}[u,u]$ represents the interaction of two colliding particles y and y' with cumulative size beyond the maximal size y_0, which merge into a virtual particle splitting instantaneously into particles all with size within the admissible range $(0,y_0)$. The daughter particles are then distributed according to $\beta_{\mathsf{s}}(y+y',y'') \geq 0$. For a more detailed physical, mathematical, and numerical discussion of this scattering phenomenon, we refer to [7], [12], [18], [25], [26], [28].

The movement of the particles is controlled by the size dependent diffusion coefficients $d(y) > 0$.
For simplicity we refrain from taking into account time dependent and spatially inhomogeneous diffusion coefficients and kernels as it has been done in earlier works [5], [6], [28].

The equations, as stated above, do not consider particle sources or sinks and thus, the total mass of all particles is expected to be preserved during time evolution. From a mathematical viewpoint, this is expressed by the formula

$$\int_\Omega \int_Y y\, u(t,y,x)\ \mathrm{d}y\mathrm{d}x = \int_\Omega \int_Y y\, u^0(y,x)\ \mathrm{d}y\mathrm{d}x\ , \quad t \geq 0\ , \tag{3}$$

which, indeed, is valid under suitable assumptions on the kernels, as we shall see.

Continuous coagulation-fragmentation equations including diffusion have not attracted much attention so far. It is their discrete counterpart which has been the object of several papers (see [16] for a comprehensive list of references for that issue). Nevertheless, the equations that are obtained from (1) by putting $P \equiv 1$ (implying $Q \equiv 0$ according to (2)) in the case $y_0 = \infty$ are investigated in [15]. There it is shown that a careful study of the reaction terms allows a treatment of the problem in the space $L_1(\Omega \times (0,\infty))$ for general diffusion coefficients $d(y)$ by using weak and strong compactness methods. Provided the kernels satisfy some suitable growth and additional structural conditions, global-in-time existence of weak solutions is proven and also their large-time behaviour is investigated in a particular situation. Subsequently, the global existence result has been improved in [19] in that also less restrictive structural conditions for the kernels have to be imposed. However, neither of these papers provides uniqueness nor conservation of mass, in general, due to the low regularity of the solutions. At this point we also refer to [9] and the references therein for probabilistic interpretations and approaches to equations of type (1). A completely different approach is chosen in [5] and [6] for the full space problem $\Omega = \mathbb{R}^n$ (with $P \equiv 1$ and $y_0 = \infty$). There, the basic idea is to treat the problem as a semilinear evolution equation of the form

$$\dot{u} + Au = f(u)\ , \quad t > 0\ , \qquad u(0) = u^0\ , \tag{4}$$

where the operator $A := -d(\cdot)\Delta_x$ acts on Banach-space-valued functions. In this reformulated form, the general semigroup-theory applies. This approach remedies the lack of regularity and thus guarantees uniqueness of strong solutions preserving the total mass. The price to be paid is that a local existence theorem is obtained only, in general. Additional, restrictive assumptions — such as particle size independent diffusion coefficients — have to be made to guarantee global existence. Semigroup-theory is also used in [28] for equations (1) with $y_0 \in (0,\infty)$. Again, local existence and uniqueness of strong solutions is shown in the space $L_p(\Omega, L_2((0,y_0)))$, where the physically somewhat artificial state space $L_2((0,y_0))$ is needed for some delicate interpolation results of Banach-space-valued L_p-spaces involving boundary conditions, and is also due to the lack of general generation results for analytic semigroups in a Banach-space-valued setting.

In the present paper we give a simpler approach than the ones in [5], [6], [28], which also ensures global existence for small initial values. More precisely, we use again semigroup methods in order to attack equations (1), but we change the order of the variables x and y, that is, we consider (1) as a problem of the form (4) but in the space $L_1((0,y_0), L_p(\Omega))$ instead of $L_p(\Omega, L_1((0,y_0)))$. We thus interpret $-A$ rather as a parameter-dependent family of generators than as generator in a vector-valued framework. It turns out that this change of the viewpoint not only allows a treatment of the problem in the more natural space $L_1((0,y_0))$, but also simplifies the proof of analyticity of the semigroup corresponding to the operator $-A = d(\cdot)\Delta_x$ considerably. Moreover, there is no need anymore to interpolate vector-valued L_p-spaces with boundary conditions. In addition, the well-known regularizing effects of the Laplace operator are easily carried over to our parameter-dependent situation. Based on these properties, a suitable choice of the function space setting allows then to consider initial values possessing only little regularity. We prove local existence and uniqueness of smooth solutions to (1), which are non-negative and preserve the total mass. Furthermore, we prove global existence for small initial values in the absence of the linear fragmentation terms.

It might be of interest that our approach also allows to cover the discrete coagulation-fragmentation equation with diffusion. Recall that in this situation the particle size only takes values in the positive integers and that the integrals appearing in the definition of the operator $L[u]$ are replaced by sums (or series). The only difference in the following would be to take the counting measure instead of the Lebesgue measure (with respect to the particle size). We point out that, for the case $y_0 < \infty$, the discrete analogue of our existence result (see Theorem 7) improves [27, Thm.1] slightly. For the case $y_0 = \infty$ we refer to [29] for comparable results. However, in order not to overload this paper we refrain from considering the discrete equation.

Of course, (1) is only a relatively simple mathematical model for rather complex physical processes. In particular, we restrict ourselves to the case of Fickian diffusion without taking into consideration mutual influences of particles of different sizes resulting in cross diffusion phenomena. A good realistic model for the diffusion effects is still lacking. In [6] a first attempt is made to jus-

tify the diffusive equations by considering the whole system as a mixture of (uncountably many) fluids and by taking cross diffusion into account as well, resulting in an additional coupling with the Navier-Stokes equations for the carrier fluid. One of the main results of this paper is the global existence assertion of Theorem 16. With our present day mathematical tools such a theorem cannot be obtained if the three dimensional Navier-Stokes equations are involved or continuous coupling due to cross diffusion occurs.
We also point out that we consider uniform dynamics only, by restricting ourselves to kernels which are independent of space and time. As already mentioned, this is done for simplicity. It is a not too difficult technical exercise to extend our results to nonuniform situations.

In the sequel, we denote by $Y := (0, y_0)$ the admissible range for the particle size. Let us emphasize again that Y may be bounded or not. Throughout this paper, the following hypotheses are supposed to be satisfied:

(H_1) K is a non-negative symmetric function belonging to $L_\infty(Y \times Y)$ and P and Q are non-negative and symmetric functions belonging to $L_\infty(\Xi)$, where

$$\Xi := \{(y, y') \in Y \times Y\,;\, y + y' \in Y\}\ ,$$

such that

$$0 \le P(y, y') + Q(y, y') \le 1 \quad \textit{for a.e. } (y, y') \in \Xi\ .$$

(H_2) γ is a measurable function from $\{(y, y')\ ;\ 0 < y' < y < y_0\}$ into $\mathbb{R}^+$ such that there exists $m_\gamma > 0$ with

$$\int_0^y \gamma(y, y')\ \mathrm{d}y' \le m_\gamma \quad \textit{for a.e. } y \in Y\ .$$

(H_3) β_{c} is a non-negative measurable function on $\{(y, y')\ ;\ 0 < y' < y < y_0\}$ such that

$$Q(y, y') \left(\int_0^{y+y'} y'' \beta_{\mathsf{c}}(y + y', y'')\ \mathrm{d}y'' - y - y' \right) = 0 \quad \textit{for a.e. } (y, y') \in \Xi\ ,$$

and there exists $m_{\mathsf{c}} > 0$ with

$$Q(y, y') \int_0^{y+y'} \beta_{\mathsf{c}}(y + y', y'')\ \mathrm{d}y'' \le m_{\mathsf{c}} \quad \textit{for a.e. } (y, y') \in \Xi\ .$$

(H_4) β_{s} is a measurable function from $(y_0, 2y_0) \times (0, y_0)$ into $\mathbb{R}^+$ such that

$$\int_0^{y_0} y'' \beta_{\mathsf{s}}(y + y', y'')\ \mathrm{d}y'' = y + y' \quad \textit{for a.e. } y + y' \in (y_0, 2y_0)\ ,$$

and there exists $m_{\mathsf{s}} \geq 2$ with

$$\int_0^{y_0} \beta_{\mathsf{s}}(y+y', y'')\, \mathrm{d}y'' \leq m_{\mathsf{s}} \quad \textit{for a.e. } y+y' \in (y_0, 2y_0)\ .$$

Example 1 *Observe that these hypotheses are satisfied provided that the splitting of particles is subject to a power-law breakup, that is, if the fragmentation kernels are of the form*

$$\begin{aligned}
\gamma(y,y') &:= h\, y^{\alpha-\xi-1}\, (y')^{\xi}\ , & 0 < y' < y < y_0\ ,\\
\beta_{\mathsf{c}}(y,y') &:= (\zeta+2)\, y^{-1-\zeta}\, (y')^{\zeta}\ , & 0 < y' < y < y_0\ ,\\
\beta_{\mathsf{s}}(y,y') &:= (\nu+2)\, y_0^{-2-\nu}\, y\, (y')^{\nu}\ , & 0 < y' < y_0 \leq y < 2y_0\ ,
\end{aligned}$$

with $h > 0$, $0 \geq \xi, \zeta, \nu > -1$, and $\alpha \geq 0$ if $y_0 < \infty$ and $\alpha := 0$ otherwise. In the latter case, β_{s} vanishes, of course. Let us point out that these kernels are more general than those considered in [28, Ex.5.13] for the case $y_0 < \infty$. This is due to the choice of $L_1((0, y_0))$ as state space instead of $L_2((0, y_0))$.

2 The Diffusion Semigroup in L_1

In the following, we use c for various constants, which may differ from occurrence to occurrence, but are always independent of the free variables. For $a, b \in \mathbb{R}$, we put $a \vee b := \max\{a, b\}$.

Let E_0 and E_1 be Banach spaces. Then $\mathcal{L}(E_1, E_0)$ consists of all linear and bounded operators from E_1 into E_0, endowed with the usual operator norm. We put $\mathcal{L}(E_0) := \mathcal{L}(E_0, E_0)$. By $\mathcal{L}^2(E_1, E_0)$ we denote the set of all continuous bilinear maps from $E_1 \times E_1$ into E_0. We write $A \in \mathcal{H}(E_0)$ if $-A$, considered as a linear (and usually unbounded) operator in E_0, is the generator of an analytic semigroup $\{e^{-tA}\,;\, t \geq 0\}$ on E_0. It is a contraction semigroup provided $\|e^{-tA}\|_{\mathcal{L}(E_0)} \leq 1$ for each $t \geq 0$. Furthermore, we use the notation $\mathcal{H}(E_1, E_0) := \mathcal{H}(E_0) \cap \mathcal{L}(E_1, E_0)$. It is known that $\mathcal{H}(E_1, E_0)$ is an open subset of $\mathcal{L}(E_1, E_0)$ (cf. [4, Thm.I.1.3.1]). Finally, if E_0 is a Banach space ordered by a closed positive cone E_0^+, the semigroup is said to be positive if $e^{-tA}(E_0^+) \subset E_0^+$ for each $t \geq 0$.

The purpose of this section is to prove that, for a given suitably bounded function d from Y into $(0, \infty)$,

$$-d(\cdot)\Delta \in \mathcal{H}\Big(L_1(Y, L_p(\Omega))\Big)\ , \quad 1 \leq p < \infty\ .$$

To this end, we introduce further notation. We denote by $H_p^\mu := H_p^\mu(\Omega)$ the usual Bessel potential space of order $\mu \geq 0$ and integrability index $p \in (1,\infty)$, and we put $L_p := L_p(\Omega)$ for $p \in [1,\infty]$ so that $H_p^0 = L_p$ for $1 < p < \infty$. For convenience we set $H_p^0 := L_p$ for $p \in \{1,\infty\}$. By L_p^+ we denote the positive cone of L_p, that is, the set of all elements in L_p which are non-negative almost everywhere. Recall that

$$H_p^\mu \hookrightarrow H_q^\alpha \ , \quad q > p \ , \quad \mu - n/p > \alpha - n/q \ ,$$

where $\hookrightarrow$ means continuous embedding. Furthermore, we define

$$H_{p,\mathcal{B}}^\mu := \begin{cases} \Big\{ u \in H_p^\mu \, ; \, \partial_\nu u = 0 \Big\} \ , & \mu > 1 + 1/p \ , \quad 1 < p < \infty \ , \\ H_p^\mu \ , & \mathit{otherwise} \ . \end{cases}$$

Then it is known (see [23]) that, for $1 < p < \infty$,

$$\Big[L_p, H_{p,\mathcal{B}}^2 \Big]_\theta \doteq H_{p,\mathcal{B}}^{2\theta} \ , \quad 2\theta \in (0,2) \setminus \{1 + 1/p\} \ , \tag{5}$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor.
In the sequel, we denote, for $1 \leq p < \infty$, by $-\Delta_p$ the closure of the linear operator

$$-\Delta : \Big\{ u \in C^2(\bar\Omega) \, ; \, \partial_\nu u = 0 \Big\} \to C(\bar\Omega) \ , \quad u \mapsto -\Delta u \ ,$$

in L_p. It follows from [2], [21] that Δ_p is well-defined and the generator of a positive, strongly continuous analytic semigroup $\{e^{t\Delta_p} \, ; \, t \geq 0\}$ of contractions on L_p. If $p \in (1,\infty)$, then $-\Delta_p \in \mathcal{H}(H_{p,\mathcal{B}}^2, L_p)$. Also note that $H_{p,\mathcal{B}}^2 \hookrightarrow \mathsf{D}(\Delta_1)$ for $1 < p < \infty$, where $\mathsf{D}(\Delta_q)$, $1 \leq q < \infty$, denotes the domain of definition of Δ_q equipped with its graph norm, and that

$$\Delta_1 \supset \Delta_p \qquad \mathit{and} \qquad e^{t\Delta_p} = e^{t\Delta_1} \,|_{L_p} \ , \quad t \geq 0 \ , \quad 1 < p < \infty \ .$$

Furthermore, for any $T > 0$,

$$\| e^{t\Delta_1} \|_{\mathcal{L}(L_p,L_q)} \leq c(T) \, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \ , \quad 0 < t \leq T \ , \quad 1 \leq p < q \leq \infty \ ,$$

and

$$\| \Delta_1 e^{t\Delta_1} \|_{\mathcal{L}(L_p)} \leq c(T) \, t^{-1} \ , \quad 0 < t \leq T \ , \quad 1 \leq p < \infty \ .$$

By interpolating according to (5) we thus obtain

$$\| e^{t\Delta_1} \|_{\mathcal{L}(L_p, H_{q,\mathcal{B}}^\alpha)} \leq c(T) \, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}} \ , \quad 0 < t \leq T \ , \tag{6}$$

for $\alpha \in [0,2] \setminus \{1 + 1/q\}$ and $1 \leq p \leq q < \infty$, where $q > 1$. Moreover,

$$\| e^{t\Delta_1} \|_{\mathcal{L}(H_{p,\mathcal{B}}^\alpha, H_{p,\mathcal{B}}^\mu)} \leq c(T) \, t^{-\frac{\mu-\alpha}{2}} \ , \quad 0 < t \leq T \ , \tag{7}$$

provided that $1 < p < \infty$ and $0 \le \alpha \le \mu \le 2$ with $\alpha, \mu \ne 1 + 1/p$.

We then introduce, for $1 \le p \le \infty$ and $\alpha \ge 0$, the spaces

$$\mathbb{L}_p := L_1\Big(Y, L_p, (1+y)\mathrm{d}y\Big) \quad and \quad \mathbb{H}^\alpha_{p,\mathcal{B}} := L_1\Big(Y, H^\alpha_{p,\mathcal{B}}, (1+y)\mathrm{d}y\Big) \ ,$$

where $L_1(Y, E, \mu)$ consists of all functions from Y into a Banach space E, which are integrable with respect to the measure μ. The specific measure $(1+y)\mathrm{d}y$ is chosen in order to give a meaning to the total mass as well as to the total number of particles for a particle size distribution belonging to $\mathbb{L}_1$. Of course, it can be replaced by the measure $\mathrm{d}y$ if Y is bounded. By $\mathbb{L}_p^+$ we denote the positive cone of $\mathbb{L}_p$, i.e., the set of all functions $u \in \mathbb{L}_p$ such that $u(y) \in L_p^+$ for a.e. $y \in Y$. Note that $\mathbb{L}_p^+$ is closed in $\mathbb{L}_p$. Given $d \in L_\infty^+(Y)$ and

$$u \in \mathsf{D}(\mathbb{A}_p) := L_1(Y, \mathsf{D}(\Delta_p), (1+y)\mathrm{d}y)$$

for some $p \in [1, \infty)$, we set

$$(\mathbb{A}_p u)(y) := -d(y)\Delta_p u(y) \ , \quad a.e.\ y \in Y \ .$$

Evidently, $\mathsf{D}(\mathbb{A}_p)$, endowed with the graph norm, coincides (except for equivalent norms) with $\mathbb{H}^2_{p,\mathcal{B}}$, provided that $p \in (1, \infty)$.

The following theorem shows that the properties of Δ_p carry over to the operator $-\mathbb{A}_p$ provided that d is bounded from above and uniformly positive.

Theorem 2 *If $d : Y \to \mathbb{R}^+$ is a measurable function such that*

$$0 < d_\star \le d(y) \le d^\star < \infty \ , \quad a.e.\ y \in Y \ , \tag{8}$$

then $\mathbb{A}_p \in \mathcal{H}(\mathbb{L}_p)$ for $1 \le p < \infty$ and $\mathbb{A}_p \supset \mathbb{A}_q$, $1 \le p \le q < \infty$. The semigroup $\{e^{-t\mathbb{A}_p} \,;\, t \ge 0\}$ is positive and given by

$$(e^{-t\mathbb{A}_p} u)(y) = e^{td(y)\Delta_p} u(y) \ , \quad a.e.\ y \in Y \ , \quad t \ge 0 \ , \quad u \in \mathbb{L}_p \ . \tag{9}$$

It is a contraction semigroup and satisfies

$$\|e^{-t\mathbb{A}_p}\|_{\mathcal{L}(\mathbb{L}_p, \mathbb{H}^\alpha_{q,\mathcal{B}})} \le c(T)\, t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{\alpha}{2}} \ , \quad 0 < t \le T \ , \tag{10}$$

for $\alpha \in [0, 2] \setminus \{1 + 1/q\}$ and $1 \le p \le q \le \infty$, where $q \in (1, \infty)$ if $\alpha > 0$. In addition,

$$\|e^{-t\mathbb{A}_p}\|_{\mathcal{L}(\mathbb{H}^\alpha_{p,\mathcal{B}}, \mathbb{H}^\mu_{p,\mathcal{B}})} \le c(T)\, t^{-\frac{\mu - \alpha}{2}} \ , \quad 0 < t \le T \ , \tag{11}$$

for $1 < p < \infty$ and $0 \le \alpha \le \mu \le 2$ with $\alpha, \mu \ne 1 + 1/p$.

Proof. Let $1 \leq p < \infty$. Clearly, $\mathbb{A}_p$ is a densely defined closed linear operator in $\mathbb{L}_p$, since $\mathsf{D}(\Delta_p)$ is dense in L_p (see [4, Thm.V.2.4.3]) and since Δ_p is a closed linear operator in L_p. Moreover, due to $-\Delta_p \in \mathcal{H}(L_p)$, there exist $\omega_p \geq 0$ and $M_p \geq 1$ such that

$$\|(\lambda - \Delta_p)^{-1}\|_{\mathcal{L}(L_p)} \leq \frac{M_p}{1+|\lambda|} , \quad \mathsf{Re}\,\lambda \geq \omega_p .$$

Assumption (8) then implies

$$\|(\lambda - d(y)\Delta_p)^{-1}\|_{\mathcal{L}(L_p)} \leq \frac{M_p'}{1+|\lambda|} , \quad \mathsf{Re}\,\lambda \geq \omega_p' , \quad a.e.\ y \in Y ,$$

where $M_p' := M_p(1 \vee 1/d_\star)$ and $\omega_p' := \omega_p d^\star$. From this we easily derive that $[\mathsf{Re}\,\lambda \geq \omega_p']$ belongs to the resolvent set of the operator $-\mathbb{A}_p$ and that the resolvent estimate

$$\|(\lambda + \mathbb{A}_p)^{-1}\|_{\mathcal{L}(\mathbb{L}_p)} \leq \frac{M_p'}{1+|\lambda|} , \quad \mathsf{Re}\,\lambda \geq \omega_p' ,$$

is valid. Hence $\mathbb{A}_p \in \mathcal{H}(\mathbb{L}_p)$ for $1 \leq p < \infty$, due to Hille's characterization of generators of analytic semigroups. Observing that

$$\Big((\lambda + \mathbb{A}_p)^{-1}u\Big)(y) = (\lambda - d(y)\Delta_p)^{-1}u(y) , \quad a.e.\ y \in Y , \quad u \in \mathbb{L}_p ,$$

for any sufficiently large $\lambda \in \mathbb{R}$, we deduce (9) from the fact that, given any Banach space E and any $A \in \mathcal{H}(E)$, the corresponding semigroup can be represented as

$$e^{-tA}v = \lim_{k\to\infty} \left(1 + \frac{t}{k}A\right)^{-k} v \qquad in \quad E \tag{12}$$

for $v \in E$. Therefore, the semigroup generated by $-\mathbb{A}_p$ is a positive semigroup of contractions. Finally, estimates (10) and (11) are easy consequences of (6)-(9). $\square$

Next we set

$$\mathbb{P}u := \frac{1}{|\Omega|}\int_\Omega u(\cdot,x)\,\mathrm{d}x , \quad u \in \mathbb{L}_p , \quad 1 \leq p \leq \infty .$$

Clearly, $\mathbb{P} \in \mathcal{L}(\mathbb{L}_p)$ is a projection and thus, the space $\mathbb{L}_p$ has the direct sum decomposition

$$\mathbb{L}_p = \mathbb{P}(\mathbb{L}_p) \oplus (1-\mathbb{P})(\mathbb{L}_p) . \tag{13}$$

The next proposition shows that the operators $e^{-t\mathbb{A}_p}, t \geq 0$, are decomposed according to (13) and it characterizes its parts in $\mathbb{P}(\mathbb{L}_p) = L_1(Y,(1+y)\mathrm{d}y)$ and $\mathbb{L}_p^\bullet := (1-\mathbb{P})(\mathbb{L}_p)$, respectively.

Proposition 3 *Let (8) be satisfied. For $1 \le p < \infty$ and $t \ge 0$, the spaces $\mathbb{P}(\mathbb{L}_p)$ and $\mathbb{L}_p^\bullet$ are both invariant under $e^{-t\mathbb{A}_p}$. Moreover, $e^{-t\mathbb{A}_p}u = u$ for each $u \in \mathbb{P}(\mathbb{L}_p)$ and there exists $\omega_0 > 0$ such that, for $1 < p < q \le \infty$ and some $M := M(p,q) > 0$,*

$$\|e^{-t\mathbb{A}_p} |_{\mathbb{L}_p^\bullet} \|_{\mathcal{L}(\mathbb{L}_p^\bullet,\mathbb{L}_q^\bullet)} \le M\, e^{-\omega_0 t}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \ , \quad t > 0 \ . \tag{14}$$

Proof. Since 1 is an eigenvector to the eigenvalue 0 for Δ_p, $1 \le p < \infty$, it follows from [2, Thm.12.1] and [14, Thm.III.6.17] that L_p has a decomposition $L_p = \mathbb{R}\cdot 1 \oplus L_p^\bullet$, which also decomposes Δ_p into $\Delta_p = 0 \oplus \Delta_p^\bullet$, where we put $\Delta_p^\bullet := \Delta_p |_{\mathsf{D}(\Delta_p)\cap L_p^\bullet}$. Thereby, we have $\mathbb{R}\cdot 1 = \mathbb{P}(L_p)$ and $L_p^\bullet = (1-\mathbb{P})(L_p)$. Moreover, denoting by $\sigma(\Delta_p^\bullet)$ the spectrum of the operator $\Delta_p^\bullet$, there exists $\omega > 0$ such that

$$\sigma(\Delta_p^\bullet) = \sigma(\Delta_p)\setminus\{0\} = \sigma(\Delta_1)\setminus\{0\} \subset [\mathsf{Re}\, z \le -\omega] \ , \tag{15}$$

since Δ_1 has a compact resolvent. Observing that

$$(\lambda - \Delta_p^\bullet)^{-1} = (\lambda - \Delta_p)^{-1} |_{L_p^\bullet} \ , \quad \lambda > 0 \ ,$$

the representation formula (12) implies that $\mathbb{R}\cdot 1$ and $L_p^\bullet$ are both invariant under $e^{t\Delta_p}$, that $e^{t\Delta_p}u = u$ for $u \in \mathbb{R}\cdot 1$ and $t \ge 0$, and that $\{e^{t\Delta_p} |_{L_p^\bullet} ;\, t \ge 0\}$ is an analytic semigroup in $L_p^\bullet$ with generator $\Delta_p^\bullet$, i.e. $e^{t\Delta_p} |_{L_p^\bullet} = e^{t\Delta_p^\bullet}, t \ge 0$. From (15) we deduce that

$$\|e^{t\Delta_p^\bullet}\|_{\mathcal{L}(L_p^\bullet)} \le M\, e^{-\omega t} \ , \quad t \ge 0 \ , \quad 1 \le p < \infty \ , \tag{16}$$

with $M := M(p) \ge 1$. Since $-\Delta_p^\bullet \in \mathcal{H}(L_p^\bullet)$ it follows that

$$\limsup_{t\to 0+} t\, \|\Delta_p^\bullet\, e^{t\Delta_p^\bullet}\|_{\mathcal{L}(L_p^\bullet)} < \infty \ .$$

Hence, there are $N := N(p) \ge 1$ and $\tau > 0$ such that

$$\|\Delta_p^\bullet\, e^{t\Delta_p^\bullet}\|_{\mathcal{L}(L_p^\bullet)} \le N\, t^{-1} \ , \quad 0 < t \le \tau \ .$$

By using the semigroup property, this estimate and (16) give

$$\|\Delta_p^\bullet\, e^{t\Delta_p^\bullet}\|_{\mathcal{L}(L_p^\bullet)} \le c\, e^{-\omega' t}\, t^{-1} \ , \quad t > 0 \ , \quad 1 \le p < \infty \ , \tag{17}$$

where $\omega' \in (0,\omega)$ and $c := c(p,\omega) > 0$. Assume now that $1 < p < q \le \infty$ and that $\mu := n(1/p - 1/q)/2 < 1$. Then

$$\|u\|_{L_q} \le c\, \|u\|_{H_p^2}^{\mu}\, \|u\|_{L_p}^{1-\mu} \ , \quad u \in H_p^2 \ ,$$

by the Gagliardo-Nierenberg inequality (e.g. [13, Thm.10.1]). Therefore, noticing that

$$e^{t\Delta_p^\bullet}u \in \mathsf{D}(\Delta_p^\bullet) = H_{p,\mathcal{B}}^2 \cap L_p^\bullet \ , \quad t > 0 \ , \quad u \in L_p^\bullet \ ,$$

we derive from (16) and (17) that

$$\|e^{t\Delta_p^\bullet}\|_{\mathcal{L}(L_p^\bullet,L_q^\bullet)} \le c\, e^{-\omega' t}\, t^{-\mu}\ , \quad t>0\ . \tag{18}$$

The semiflow property guarantees that this estimate remains valid also for $\mu \ge 1$. We now easily infer from (9) that (13) decomposes $e^{-t\mathbb{A}_p}$ into

$$e^{-t\mathbb{A}_p} = 1_{L_1(Y,(1+y)\mathrm{d}y)} \oplus \left(e^{-t\mathbb{A}_p}\,|_{\mathbb{L}_p^\bullet}\right)\ , \quad t\ge 0\ , \quad 1\le p<\infty\ ,$$

and that

$$(e^{-t\mathbb{A}_p}u)(y) = e^{td(y)\Delta_p^\bullet}u(y)\ , \quad a.e.\ y\in Y\ , \quad t\ge 0\ , \quad u\in\mathbb{L}_p^\bullet\ .$$

Thus (18) and (8) imply, for $\omega_0 := d_\star\omega' > 0$, that

$$\|e^{-t\mathbb{A}_p}\,|_{\mathbb{L}_p^\bullet}\,\|_{\mathcal{L}(\mathbb{L}_p^\bullet,\mathbb{L}_q^\bullet)} \le c\, e^{-\omega_0 t}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\ , \quad t>0\ ,$$

provided $1<p<q\le\infty$. $\square$

3 Existence and Uniqueness

We now turn to the well-posedness of problem (1). In this section we show that this problem admits a unique maximal strong solution. The questions of positivity, conservation of mass, and global existence are postponed to sections 4 and 5, respectively.

In what follows, we always assume that hypotheses $(H_1)-(H_4)$ and (8) are satisfied.
Let $p\in[1,\infty)$ and observe that the pointwise product $L_{2p}\times L_{2p}\to L_p$ is a multiplication, that is, a continuous bilinear map of norm at most one. Thus, hypotheses $(H_1)-(H_4)$ readily imply that

$$L_{\mathsf{b}}\in\mathcal{L}(\mathbb{L}_p) \quad \textit{and} \quad G := L_{\mathsf{c}} + L_{\mathsf{s}} \in \mathcal{L}^2(\mathbb{L}_{2p},\mathbb{L}_p)\ . \tag{19}$$

Hence, we may rewrite (1) as a Cauchy problem in $\mathbb{L}_p$ of the form

$$\dot u + \mathbb{A}_p u = L_{\mathsf{b}}[u] + G[u,u]\ , \quad t>0\ , \quad u(0)=u^0\ . \tag{20}$$

Let us first state what we mean by a solution to problem (20). Assume that $J\subset\mathbb{R}^+$ is a perfect interval containing 0 and put $\dot J := J\setminus\{0\}$. A function

$u \in C(J, \mathbb{L}_p)$ is said to be a *mild $\mathbb{L}_p$-solution* to (20) on J provided that u solves the integral equation

$$u(t) = e^{-t\mathbb{A}_p} u^0 + \int_0^t e^{-(t-s)\mathbb{A}_p} \Big(L_{\mathsf{b}}[u(s)] + G[u(s), u(s)] \Big) \, \mathrm{d}s \ , \quad t \in J \ . \tag{21}$$

If, in addition, $u \in C^1(\dot{J}, \mathbb{L}_p) \cap C(\dot{J}, \mathsf{D}(\mathbb{A}_p))$ then u is a *(strong) $\mathbb{L}_p$-solution* to (20) on J. Recall that $\mathsf{D}(\mathbb{A}_p) = \mathbb{H}^2_{p,\mathcal{B}}$ for $p \in (1, \infty)$.

Given a Banach space E and $\mu \in \mathbb{R}$, we denote by $BC_\mu(\dot{J}, E)$ the Banach space of all functions $u : \dot{J} \to E$ such that $\big(t \mapsto t^\mu u(t)\big)$ is bounded and continuous from $\dot{J}$ into E, equipped with the norm

$$u \mapsto \|u\|_{BC_\mu(\dot{J},E)} := \sup_{t \in \dot{J}} t^\mu \, \|u(t)\|_E \ .$$

We write $C_\mu(\dot{J}, E)$ for the closed linear subspace thereof consisting of all u satisfying $t^\mu u(t) \to 0$ in E as $t \to 0$. Note that $C_\nu((0,T], E) \hookrightarrow C_\mu((0,T], E)$ for $\nu \le \mu$ and $T > 0$.

For convenience, we set $\mathbb{A} := \mathbb{A}_1$ and $U(t) := e^{-t\mathbb{A}}$, $t \ge 0$, and we consider $\{U(t)\,;\, t \ge 0\}$ as semigroup in any of the spaces $\mathbb{L}_p$, $p \in [1, \infty)$, since no confusion will arise in the sequel thanks to Theorem 2. Furthermore, given $u \in L_1(\dot{J}, \mathbb{L}_1)$, we put

$$U \star u(t) := \int_0^t U(t-s)\, u(s) \, \mathrm{d}s \ , \quad t \in \dot{J} \ ,$$

whenever these integrals exist. For the following, let $T > 0$ be arbitrary and set $J := [0, T]$.

Proposition 4 *Let $1 \le p \le q \le \infty$ and $\alpha \in [0,2] \setminus \{1 + 1/q\}$ be such that $n(1/p - 1/q)/2 + \alpha/2 < 1$ and either $q \in (1, \infty)$ or $\alpha = 0$. Then, for $\mu < 1$,*

$$\Big(u \mapsto U \star L_{\mathsf{b}}[u]\Big) \in \mathcal{L}\Big(C_\mu(\dot{J}, \mathbb{L}_p), C_{\mu + \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{\alpha}{2} - 1}(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})\Big) \tag{22}$$

and

$$\Big(u \mapsto U \star G[u,u]\Big) \in \mathcal{L}^2\Big(C_{\mu/2}(\dot{J}, \mathbb{L}_{2p}), C_{\mu + \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{\alpha}{2} - 1}(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})\Big) \ . \tag{23}$$

Proof. Due to (19) it suffices to prove that

$$\Big(u \mapsto U \star u\Big) \in \mathcal{L}\Big(C_\mu(\dot{J}, \mathbb{L}_p), C_{\mu + \zeta - 1}(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})\Big)$$

for $\zeta := n(1/p - 1/q)/2 + \alpha/2$. But (10) implies that, for $u \in C_\mu(\dot{J}, \mathbb{L}_p)$ and $t \in \dot{J}$,

$$
\begin{aligned}
\|U \star u(t)\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} &\leq c(T) \int_0^t (t-s)^{-\zeta}\, s^{-\mu}\, \mathrm{d}s\, \|u\|_{BC_\mu((0,t),\mathbb{L}_p)} \\
&= c(T)\, t^{1-\zeta-\mu}\, \mathsf{B}(1-\zeta, 1-\mu)\, \|u\|_{BC_\mu((0,t),\mathbb{L}_p)}\ ,
\end{aligned}
$$

where B denotes the beta function. Therefore, $U \star u \in C_{\mu+\zeta-1}(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})$ since

$$
\|u\|_{BC_\mu((0,t),\mathbb{L}_p)} \to 0 \quad as \quad t \to 0\ .
$$

Thus the assertion follows. $\square$

Remark 5 *It is obvious that the norms of the maps in* (22) *and* (23) *are increasing with respect to length of the interval J, that is, with respect to $T > 0$.*

Proposition 6 *Let $1 \leq p \leq q < \infty$, $\alpha \in [0,2] \setminus \{1 + 1/q\}$ and assume that $n(1/p - 1/q)/2 + \alpha/2 < 1$, where either $\alpha = 0$ and $p < q$ or $\alpha > 0$ and $q \in (1, \infty)$. Then, for $u^0 \in \mathbb{L}_p$,*

$$
Uu^0 := \Big(t \mapsto U(t)u^0\Big) \in C_{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{\alpha}{2}}(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})\ .
$$

Proof. From (5) and the Riesz-Thorin theorem [24, Thm.1.18.4] we infer that

$$
\Big[\mathbb{L}_q, \mathbb{H}^2_{q,\mathcal{B}}\Big]_\theta \doteq \mathbb{H}^{2\theta}_{q,\mathcal{B}}\ , \quad 2\theta \neq 1 + 1/q\ . \tag{24}
$$

Theorem 2 and [4, Thm.V.2.1.3] yield that the $\mathbb{H}^\alpha_{q,\mathcal{B}}$-realization of $\mathbb{A}$ belongs to $\mathcal{H}(\mathbb{H}^\alpha_{q,\mathcal{B}})$. In particular, we have

$$
\|\mathbb{A}\, U(t)\|_{\mathcal{L}(\mathbb{H}^\alpha_{q,\mathcal{B}})} \leq c(T)\, t^{-1}\ , \quad t \in \dot{J}\ .
$$

Given $t, t+h \in \dot{J}$ with $h > 0$, it then follows from [20, Thm.1.2.4] and (10) that

$$
\begin{aligned}
\|U(t+h)u^0 - U(t)u^0\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} &= \|\mathbb{A} \int_0^h U(s)U(t)u^0\ \mathrm{d}s\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} \\
&\leq \|\mathbb{A}\, U(t)\|_{\mathcal{L}(\mathbb{H}^\alpha_{q,\mathcal{B}})} \int_0^h \|U(s)\|_{\mathcal{L}(\mathbb{L}_p,\mathbb{H}^\alpha_{q,\mathcal{B}})}\ \mathrm{d}s\ \|u^0\|_{\mathbb{L}_p} \\
&\leq c(T)\, t^{-1}\, h^{1-\zeta}\, \|u^0\|_{\mathbb{L}_p}\ ,
\end{aligned}
$$

for $\zeta := n(1/p - 1/q)/2 + \alpha/2$. Hence Uu^0 is continuous on $\dot{J}$ with values in $\mathbb{H}^\alpha_{q,\mathcal{B}}$. From this we infer that $Uu^0 \in BC_\zeta(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})$. Since $H^\alpha_{q,\mathcal{B}}$ is dense in L_p, it follows from [4, Thm.V.2.4.3] that $\mathbb{H}^\alpha_{q,\mathcal{B}}$ is dense in $\mathbb{L}_p$. Thus, given any $\varepsilon > 0$, there exists $v \in \mathbb{H}^\alpha_{q,\mathcal{B}}$ such that

$$\|u^0 - v\|_{\mathbb{L}_p} \le \varepsilon / \sup_{t\in\dot{J}} t^\zeta \, \|U(t)\|_{\mathcal{L}(\mathbb{L}_p,\mathbb{H}^\alpha_{q,\mathcal{B}})} \ .$$

Therefore, we have

$$t^\zeta \, \|U(t)u^0\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} \le t^\zeta \, \|U(t)\|_{\mathcal{L}(\mathbb{H}^\alpha_{q,\mathcal{B}})} \, \|v\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} + \varepsilon \ , \quad t \in \dot{J} \ .$$

The assertion is then a consequence of (11) and $\zeta > 0$, since $\varepsilon > 0$ was arbitrary. $\square$

We can prove now the existence and uniqueness of maximal solutions to (20).

Theorem 7 *Assume that $p \in (n/2, \infty)$ and $p \ge 1$. Then, given any initial value $u^0 \in \mathbb{L}_p$, problem (20) possesses a unique maximal $\mathbb{L}_p$-solution $u := u(\cdot; u^0)$ on $J(u^0)$ such that*

$$t^{n/4p} \, \|u(t)\|_{\mathbb{L}_{2p}} \to 0 \quad as \quad t \to 0+ \ .$$

Here $J(u^0)$ is an open interval in $\mathbb{R}^+$. In addition,

$$u \in C^1(\dot{J}(u^0), \mathbb{L}_q) \cap C(\dot{J}(u^0), \mathbb{H}^2_{q,\mathcal{B}}) \ , \quad q \in (1,\infty) \ .$$

If $t^+ := \sup\, J(u^0) < \infty$, then

$$\sup_{t^+/2 < t < t^+} \|u(t)\|_{\mathbb{L}_q} = \infty \ , \quad q > n/2 \quad with \quad q \ge 1 \ . \tag{25}$$

Proof. (i) Let $T_0 > 0$ be arbitrary and define $X_T := C_\mu((0,T], \mathbb{L}_{2p})$ for $T \in (0, T_0]$, where $\mu := n/4p$. Proposition 6 yields $Uu^0 \in X_T$. Consequently, Proposition 4 and Remark 5 imply that there exists a constant $\kappa := \kappa(T_0) > 0$ such that the map $F : X_T \to X_T$, given by

$$F(u) := Uu^0 + U \star \Big(G[u,u] + L_{\mathsf{b}}[u] \Big) \ , \quad u \in X_T \ ,$$

satisfies

$$\|F(u) - F(v)\|_{X_T} \le \kappa \, (T + \|u\|_{X_T} + \|v\|_{X_T}) \, \|u - v\|_{X_T} \ , \quad u, v \in X_T \ , \tag{26}$$

and

$$\|F(u) - Uu^0\|_{X_T} \le \kappa \, (T + \|u\|_{X_T}) \, \|u\|_{X_T} \ , \quad u \in X_T \ . \tag{27}$$

Set $R := 1/16\kappa$ and choose $T \in (0, T_0]$ such that $T < 1/2\kappa$ and $\|Uu^0\|_{X_T} \le R$. Let B_T denote the closed ball in X_T centered at Uu^0 with radius $3R$. Then, since $\|u\|_{X_T} \le 4R$ for each $u \in B_T$, (26) and (27) imply that $F : B_T \to B_T$

is a contraction. Therefore, there exists a unique $\bar{u} \in B_T$ with $F(\bar{u}) = \bar{u}$, that is, $\bar{u} \in C_\mu((0,T], \mathbb{L}_{2p})$ and

$$\bar{u} = Uu^0 + U \star \Big(G[\bar{u},\bar{u}] + L_{\mathsf{b}}[\bar{u}] \Big) \ . \tag{28}$$

Due to $\mu < 1/2$, we derive from Proposition 4 that

$$U \star G[\bar{u},\bar{u}] \in C_{2\mu-1}((0,T], \mathbb{L}_p) \hookrightarrow C_0((0,T], \mathbb{L}_p)$$

and

$$U \star L_{\mathsf{b}}[\bar{u}] \in C_{\mu-1}((0,T], \mathbb{L}_p) \hookrightarrow C_0((0,T], \mathbb{L}_p) \ .$$

Hence $\bar{u} \in C([0,T], \mathbb{L}_p) \cap C_\mu((0,T], \mathbb{L}_{2p})$ is a mild $\mathbb{L}_p$-solution to (20).

(ii) Let $q > p$ and fix $\alpha \in (n/2q, 2) \setminus \{1 + 1/q\}$ such that $n/p - n/q + \alpha < 2$. Invoking Propositions 4 and 6, equality (28) shows that there is a $\bar{\mu} > 0$ with

$$\bar{u} \in C_{\bar{\mu}}((0,T], \mathbb{H}^\alpha_{q,\mathcal{B}}) \ . \tag{29}$$

Since $\alpha > n/2q$, the pointwise product $H^\alpha_{q,\mathcal{B}} \times H^\alpha_{q,\mathcal{B}} \to H^\nu_{q,\mathcal{B}}$ is a multiplication, where $\nu > 0$ is chosen sufficiently small (see [22, Cor.4.5.2]). Therefore,

$$G \in \mathcal{L}^2(\mathbb{H}^\alpha_{q,\mathcal{B}}, \mathbb{H}^\nu_{q,\mathcal{B}}) \quad \textit{and} \quad L_{\mathsf{b}} \in \mathcal{L}(\mathbb{H}^\alpha_{q,\mathcal{B}}, \mathbb{H}^\nu_{q,\mathcal{B}}) \ ,$$

so (29) gives $h := G[\bar{u},\bar{u}] + L_{\mathsf{b}}[\bar{u}] \in C_{2\bar{\mu}}((0,T], \mathbb{H}^\nu_{q,\mathcal{B}})$. In particular, for each $\varepsilon \in (0,T)$, we have

$$h_\varepsilon := h(\cdot + \varepsilon) \in C([0, T-\varepsilon], \mathbb{H}^\nu_{q,\mathcal{B}}) \ .$$

Thus, since $\mathbb{A} \in \mathcal{H}(\mathbb{H}^2_{q,\mathcal{B}}, \mathbb{L}_q)$ due to Theorem 2, it follows from (24), [4, Thm.IV.1.5.1], and $\bar{u}(\varepsilon) \in \mathbb{H}^\alpha_{q,\mathcal{B}}$ that the linear problem

$$\dot{v} + \mathbb{A}v = h_\varepsilon(t) \ , \quad 0 < t \le T - \varepsilon \ , \qquad v(0) = \bar{u}(\varepsilon) \ , \tag{30}$$

possesses a unique solution

$$v \in C((0, T-\varepsilon], \mathbb{H}^2_{q,\mathcal{B}}) \cap C^1((0, T-\varepsilon], \mathbb{L}_q) \cap C([0, T-\varepsilon], \mathbb{L}_q) \ .$$

It coincides with $\bar{u}(\cdot + \varepsilon)$ in view of the facts that the latter is a mild solution to (30) as well and that mild solutions to linear problems are unique. This being true for every $\varepsilon \in (0,T)$, we conclude that $\bar{u} \in C((0,T], \mathbb{H}^2_{q,\mathcal{B}}) \cap C^1((0,T], \mathbb{L}_q)$, so that it is indeed a strong $\mathbb{L}_p$-solution to (20).

(iii) Clearly, we can extend $\bar{u}$ to a unique maximal solution $u = u(\cdot; u^0)$, where the maximal interval of existence, $J(u^0)$, is necessarily open in $\mathbb{R}^+$. Consider then the case that $t^+ := \sup J(u^0) < \infty$. First assume that there exists a sequence $t_j \nearrow t^+$ such that $\|u(t_j)\|_{\mathbb{L}_q} \le r < \infty$ for each $j \in \mathbb{N}$ and some $q \in (p, 2p)$. Fix $T_0 > t^+$ and let $R = R(T_0)$ denote the corresponding constant from part (i). Then, since, for $j \in \mathbb{N}$ and $0 < t \le T \le T_0$,

$$t^{n/4p} \, \|U(t)u(t_j)\|_{\mathbb{L}_{2p}} \le r \Big(\sup_{0<\tau\le T_0} \tau^{\frac{n}{2}(\frac{1}{q}-\frac{1}{2p})} \, \|U(\tau)\|_{\mathcal{L}(\mathbb{L}_q,\mathbb{L}_{2p})} \Big) \, T^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \ ,$$

we can choose $T > 0$ sufficiently small such that $\|Uu(t_j)\|_{X_T} \le R$ for $j \in \mathbb{N}$. Part (i) now shows that u exists at least on $[t_j, t_j + T]$ contradicting its maximality. Thus, for $u^0 \in \mathbb{L}_p$ we have

$$\sup_{t^+/2<t<t^+} \|u(t)\|_{\mathbb{L}_q} = \infty \ , \quad q > p \ . \tag{31}$$

Next suppose that $p > 1$ and let $(n/2 \vee 1) < \bar{q} < p$ be arbitrary. Denoting by $u_{\bar{q}}$ the unique maximal $\mathbb{L}_{\bar{q}}$-solution to the initial value $u^0 \in \mathbb{L}_p \hookrightarrow \mathbb{L}_{\bar{q}}$, whereas u still denotes the unique maximal $\mathbb{L}_p$-solution on $J(u^0)$, we obviously have $u_{\bar{q}} \supset u$. Assume that $J_{\bar{q}}$, the domain of $u_{\bar{q}}$, is a proper extension of $J(u^0)$. Then, by virtue of (31),

$$\sup_{t^+/2<t<t^+} \|u_{\bar{q}}(t)\|_{\mathbb{L}_q} = \sup_{t^+/2<t<t^+} \|u(t)\|_{\mathbb{L}_q} = \infty$$

for $q > p$, which contradicts the fact that $u_{\bar{q}} \in C(\dot{J}_{\bar{q}}, \mathbb{L}_q)$ according to part (ii). We infer that $u_{\bar{q}} = u$. Hence, by applying (31) to $u_{\bar{q}}$, we derive

$$\sup_{t^+/2<t<t^+} \|u(t)\|_{\mathbb{L}_q} = \infty \ , \quad q > (n/2 \vee 1) \ . \tag{32}$$

Therefore, we are left to prove that (32) is valid for $q = 1$ in the particular case $n = 1$. For that purpose, assume, by contradiction, that this would be false. Then, since $u \in C(J(u^0), \mathbb{L}_1) \cap C_{1/4}(\dot{J}(u^0), \mathbb{L}_2)$, there is $c_0 > 0$ with $\|u(t)\|_{\mathbb{L}_1} \le c_0$ for $t \in J(u^0)$. Choose $q \in (1, 2)$ and $\alpha \in (1/q, 1+1/q)$ arbitrarily and put $\zeta := (1 - 1/q)/2 + \alpha/2 \in (0, 1)$. From Propositions 4 and 6 we infer that $u \in C_\zeta(\dot{J}(u^0), \mathbb{H}^\alpha_{q,\mathcal{B}})$. Moreover, $(H_1) - (H_4)$ yield

$$\|L[u(t)]\|_{\mathbb{L}_1} \le c \, (1 + \|u(t)\|_{\mathbb{L}_1}) \, \|u(t)\|_{\mathbb{L}_\infty} \le c_0' \, \|u(t)\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} \ , \quad t \in \dot{J}(u^0) \ ,$$

due to the embedding $\mathbb{H}^\alpha_{q,\mathcal{B}} \hookrightarrow \mathbb{L}_\infty$. Taking (10) into account, we obtain

$$\begin{aligned} \|u(t)\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} &\le \|U(t)\|_{\mathcal{L}(\mathbb{L}_1,\mathbb{H}^\alpha_{q,\mathcal{B}})} \, \|u^0\|_{\mathbb{L}_1} + \int_0^t \|U(t-s)\|_{\mathcal{L}(\mathbb{L}_1,\mathbb{H}^\alpha_{q,\mathcal{B}})} \, \|L[u(s)]\|_{\mathbb{L}_1} \, \mathrm{d}s \\ &\le c(t^+) \, t^{-\zeta} \, \|u^0\|_{\mathbb{L}_1} + c(t^+) \int_0^t (t-s)^{-\zeta} \, \|u(s)\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} \, \mathrm{d}s \end{aligned}$$

for $t \in \dot{J}(u^0)$. Since $u \in C_\zeta(\dot{J}(u^0), \mathbb{H}^\alpha_{q,\mathcal{B}})$, we may apply the singular Gronwall inequality [4, Cor.II.3.3.2] to obtain that

$$\|u(t)\|_{\mathbb{L}_q} \le c \, \|u(t)\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} \le c(t^+) \ , \quad t^+/2 < t < t^+ \ ,$$

which is impossible according to (32). $\square$

Remarks 8 **(a)** *Let $p \in (n/2, \infty)$ with $p \geq 1$ and $v \in C_{n/4p}((0,T], \mathbb{L}_{2p})$. Then v is a mild $\mathbb{L}_p$-solution to (20) if and only if v is a strong $\mathbb{L}_p$-solution. In this case, v belongs to $C^1((0,T], \mathbb{L}_q) \cap C((0,T]), \mathbb{H}^2_{q,\mathcal{B}})$ for each $q \in (1, \infty)$.*

Proof. This is a consequence of part (ii) of the proof of the previous theorem. $\square$

(b) *Let $(n/2 \vee 1) < p < \infty$ and $u^0 \in \mathbb{L}_p$. Then the solution $u(\cdot; u^0)$ is unique among all mild $\mathbb{L}_q$-solutions v satisfying $t^{n/4q} \|v(t)\|_{\mathbb{L}_{2q}} \to 0$ for some $q \in ((n/2 \vee 1), p]$.*

Proof. This has been observed in part (iii) of the proof of Theorem 7 and the previous remark (a). $\square$

(c) *Let $p \in (n/2, \infty)$ with $p \geq 1$ and assume that $u^0 \in \mathbb{L}_p$. Then there exist $\delta > 0$ and $T := T(u^0) > 0$ such that $J(v^0) \supset [0,T]$ for each $v^0 \in \mathbb{L}_p$ with $\|u^0 - v^0\|_{\mathbb{L}_p} \leq \delta$. Moreover,*

$$u(\cdot; v^0) \to u(\cdot; u^0) \quad \textit{in} \quad C_{n/4p}((0,T], \mathbb{L}_{2p}) \quad \textit{as} \quad v^0 \to u^0 \quad \textit{in} \quad \mathbb{L}_p \ .$$

Proof. We use the notation of the proof of Theorem 7. Choose $T \in (0, T_0]$ such that $T < 1/4\kappa$ and $\|Uu^0\|_{X_T} \leq R/2$, and put $\delta := R/2\varrho$, where

$$\varrho := \sup_{0<t\leq T_0} t^{n/4p} \|U(t)\|_{\mathcal{L}(\mathbb{L}_p, \mathbb{L}_{2p})} \ .$$

Then $\|Uv^0\|_{X_T} \leq R$ provided that $v^0 \in \mathbb{L}_p$ with $\|u^0 - v^0\|_{\mathbb{L}_p} \leq \delta$. From part (i) of the proof of Theorem 7 we infer the existence of solutions $\bar{u} = u(\cdot; u^0) \mid_{[0,T]}$ and $\bar{v} = u(\cdot; v^0) \mid_{[0,T]}$, both belonging to X_T, with initial value u^0 and v^0, respectively, such that

$$\|\bar{u} - Uu^0\|_{X_T} \leq 3R \quad \textit{and} \quad \|\bar{v} - Uv^0\|_{X_T} \leq 3R \ ,$$

whence $\|\bar{u}\|_{X_T} + \|\bar{v}\|_{X_T} \leq 8R$. Since, due to (26),

$$\|\bar{u} - \bar{v}\|_{X_T} \leq \varrho \, \|u^0 - v^0\|_{\mathbb{L}_p} + \kappa \, (T + \|\bar{u}\|_{X_T} + \|\bar{v}\|_{X_T}) \, \|\bar{u} - \bar{v}\|_{X_T} \ ,$$

we thus conclude that

$$\|\bar{u} - \bar{v}\|_{X_T} \leq 4 \, \varrho \, \|u^0 - v^0\|_{\mathbb{L}_p} \ ,$$

which proves everything. $\square$

(d) *If $(n/2 \vee 1) < p < \infty$, then $u(\cdot; u^0) \in C(J(u^0), \mathbb{H}^\alpha_{p,\mathcal{B}})$ provided that $u^0 \in \mathbb{H}^\alpha_{p,\mathcal{B}}$ with $0 \leq \alpha < 2 - n/p$ and $\alpha \neq 1 + 1/p$.*

Proof. Observing that $u = u(\cdot; u^0) \in C_{n/4p}((0,T], \mathbb{L}_{2p})$ for $T \in \dot{J}(u^0)$ and that $\nu := n/2p - 1 + \alpha/2 < 0$, it follows from Proposition 4 that

$$U \star G[u,u] \in C_\nu((0,T], \mathbb{H}^\alpha_{p,\mathcal{B}}) \hookrightarrow C_0((0,T], \mathbb{H}^\alpha_{p,\mathcal{B}})$$

and

$$U \star L_{\mathsf{b}}[u] \in C_{\nu-n/4p}((0,T], \mathbb{H}^\alpha_{p,\mathcal{B}}) \hookrightarrow C_0((0,T], \mathbb{H}^\alpha_{p,\mathcal{B}}) \ .$$

Taking into account that $Uu^0 \in C(\mathbb{R}^+, \mathbb{H}^\alpha_{p,\mathcal{B}})$ since the $\mathbb{H}^\alpha_{p,\mathcal{B}}$-realization of $\mathbb{A}$ belongs to $\mathcal{H}(\mathbb{H}^\alpha_{p,\mathcal{B}})$ (see the proof of Proposition 6), the assertion follows from (21). □

4 Positivity and Conservation of Mass

We now prove that the solution $u(\cdot; u^0)$, given by Theorem 4, remains non-negative whenever it is non-negative at time zero, that is, if $u^0 \in \mathbb{L}_p^+$.

First, we need the following auxiliary lemma.

Lemma 9 *Let $q > p \geq 1$ and $\alpha \geq 0$ with $\alpha \neq 1 + 1/q$. Then $\mathbb{H}^\alpha_{q,\mathcal{B}} \cap \mathbb{L}_q^+$ is dense in $\mathbb{L}_p^+$.*

Proof. For an open subset X of $\mathbb{R}^m$ denote by $\mathcal{D}^+(X)$ the non-negative test functions on X. Clearly, $\mathcal{D}^+(\Omega)$ is dense in L_p^+. Hence, the tensor product $\mathcal{D}^+(\mathbb{R}) \otimes \mathcal{D}^+(\Omega)$ is dense in $\mathcal{D}^+(\mathbb{R}) \otimes L_p^+$. By (the proof of) [4, Prop.V.2.4.1] the latter space is dense in $\mathcal{D}^+(\mathbb{R}, L_p)$, i.e., in the space of all test functions on $\mathbb{R}$ with values in L_p^+. Standard cutting and mollification arguments show that $\mathcal{D}^+(\mathbb{R}, L_p)$ is dense in $L_1^+(\mathbb{R}, L_p)$. Consequently, $\mathcal{D}^+(\mathbb{R}) \otimes \mathcal{D}^+(\Omega)$ is dense in $L_1^+(\mathbb{R}, L_p)$. By extending the elements of $\mathbb{L}_p^+ = L_1^+(Y, L_p, (1+y)\mathrm{d}y)$ by zero outside of Y, we deduce the claimed statement. □

The proof of the positivity of the solution $u(\cdot; u^0)$ is based on the previous lemma and the continuous dependence on the initial value.

Theorem 10 *Let $p \in (n/2, \infty)$ with $p \geq 1$. Then $u(t; u^0) \in \mathbb{L}_p^+$ for $t \in J(u^0)$ provided that $u^0 \in \mathbb{L}_p^+$.*

Proof. (i) First suppose that $u^0 \in \mathbb{H}^\alpha_{p,\mathcal{B}} \cap \mathbb{L}_p^+$ with $n/p < \alpha < 2 - n/p$ and $\alpha \neq 1 + 1/p$. Remark 8(d) implies that

$$u = u(\cdot; u^0) \in C(J(u^0), \mathbb{H}^\alpha_{p,\mathcal{B}}) \hookrightarrow C(J(u^0), L_1(Y, C(\bar{\Omega}))) \ .$$

Let $T_0 \in \dot{J}(u^0)$ be arbitrary. Then, there is some constant $\omega := \omega(T_0) > 0$

such that, for $t \in [0, T_0]$,

$$\left| \int_Y K(y,y')\, u(t,y',x)\; \mathrm{d}y' \right| + \left| \int_0^y \frac{y'}{y} \gamma(y,y')\; \mathrm{d}y' \right| \le \omega \ , \quad a.e.\ y \in Y \ , \quad x \in \Omega \ .$$

For $v \in \mathbb{L}_{2p}$ and $0 \le t \le T_0$ set

$$H(t,v) := L_{\mathsf{b}}[v] + L_{\mathsf{c}}^1[v,v] + L_{\mathsf{c}}^2[v,v] + L_{\mathsf{s}}^1[v,v] - L_{\mathsf{c}}^3[v,u(t)] - L_{\mathsf{s}}^2[v,u(t)] + \omega\, v \ ,$$

so $H(t,v)(y,x) \ge 0$ for $(t,x) \in [0,T_0] \times \Omega$ and a.e. $y \in Y$, provided that v belongs to $L_1^+(Y, C(\bar{\Omega}), (1+y)\mathrm{d}y)$. Next observe that u satisfies the equation

$$\dot{u} + (\omega + \mathbb{A})u = H(t,u) \ , \quad t \in (0,T_0] \ , \qquad u(0) = u^0 \ ,$$

which can be solved by the method of successive approximation. Thus, define

$$F(v) := U_\omega u^0 + U_\omega \star H(\cdot, v) \ , \quad v \in X_T := C_{n/4p}((0,T], \mathbb{L}_{2p}) \ ,$$

where $U_\omega(t) := e^{-\omega t} U(t)$, $t \ge 0$. Then one shows, analogously to the proof of Theorem 7, that F is a contraction from a suitable ball B_T in X_T centered at Uu^0 into itself. By making T smaller if necessary we also may assume that $u \in B_T$. Therefore, the sequence (u_j), determined by

$$u_0 := u^0 \ , \qquad u_{j+1} := F(u_j) \ , \quad j \in \mathbb{N} \ ,$$

converges to u in X_T. In particular, $u_j(t) \to u(t)$ in $\mathbb{L}_{2p}$ for $t \in (0,T]$. Since $\{U(t)\,;\, t \ge 0\}$ is a positive semigroup according to Theorem 2, we see by induction that $u_j(t) \in \mathbb{L}_{2p}^+$ for $0 < t \le T$ and $j \in \mathbb{N}$. So $u(t) \in \mathbb{L}_{2p}^+$ since $\mathbb{L}_{2p}^+$ is closed. Let $T^\star \le T_0$ denote the maximal time for which u is positive on $[0, T^\star]$. Then $T^\star = T_0$ since, otherwise, repeating the above arguments for problem

$$\dot{v} + (\omega + \mathbb{A})v = H(t + T^\star, v) \ , \quad t \in (0, T_0 - T^\star] \ , \qquad v(0) = u(T^\star) \ ,$$

would lead to a contradiction. $T_0 \in J(u^0)$ being arbitrary, we deduce that $u(t) \in \mathbb{L}_p^+$ for all $t \in J(u^0)$.

(ii) Finally, for arbitrary $p \in (n/2, \infty)$ with $p \ge 1$ and $u^0 \in \mathbb{L}_p^+$, we can use Remark 8(c), Lemma 9, and part (i) to conclude that $u(t) \in \mathbb{L}_p^+$ for all $t \in J(u^0)$. This proves the claim. $\square$

We now substantiate the intuitive guess that the total mass is preserved during time. This simple observation is based on the following identities, which are easy consequences of hypotheses $(H_1) - (H_4)$ and the Fubini theorem (see [25, Lem.2.6]).

Lemma 11 *Given $v \in L_1(Y,(1+y)\mathrm{d}y)$ and $k=0,1$, the following identities are valid:*

$$\begin{aligned}
\int_0^{y_0} y^k\, L_{\mathsf{b}}[v](y)\ \mathrm{d}y \ &= \ \int_0^{y_0}\int_0^y \left\{(y')^k - y'\, y^{k-1}\right\}\gamma(y,y')\ \mathrm{d}y'\ v(y)\ \mathrm{d}y\ ,\\
\int_0^{y_0} y^k\, L_{\mathsf{c}}[v,v](y)\ \mathrm{d}y \ &= \ \frac{1}{2}\int_0^{y_0}\int_0^{y_0-y} \Phi_{\mathsf{c}}^{(k)}(y,y')\, K(y,y')\, v(y')\, v(y)\ \mathrm{d}y'\mathrm{d}y\ ,\\
\int_0^{y_0} y^k\, L_{\mathsf{s}}[v,v](y)\ \mathrm{d}y \ &= \ \frac{1}{2}\int_0^{y_0}\int_{y_0-y}^{y_0} \Phi_{\mathsf{s}}^{(k)}(y,y')\, K(y,y')\, v(y')\, v(y)\, \mathrm{d}y'\mathrm{d}y\ ,
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{\mathsf{c}}^{(k)}(y,y') := P(y,y')\ &\left((y+y')^k - y^k - (y')^k\right)\\
&+ Q(y,y')\left(\int_0^{y+y'} (y'')^k\, \beta_{\mathsf{c}}(y+y',y'')\ \mathrm{d}y'' - y^k - (y')^k\right)
\end{aligned}$$

and

$$\Phi_{\mathsf{s}}^{(k)}(y,y') := \int_0^{y_0} (y'')^k\, \beta_{\mathsf{s}}(y+y',y'')\ \mathrm{d}y'' - y^k - (y')^k\ .$$

Theorem 12 *Let $p \in (n/2,\infty)$ with $p \ge 1$. Then, for each $u^0 \in \mathbb{L}_p$,*

$$\int_\Omega\int_Y y\, u(t,y,x)\ \mathrm{d}y\mathrm{d}x = \int_\Omega\int_Y y\, u^0(y,x)\ \mathrm{d}y\mathrm{d}x\ ,\quad t\in J(u^0)\ .$$

Proof. Lemma 11 and $(H_3),(H_4)$ yield that, for $v \in L_1(Y,(1+y)\mathrm{d}y)$,

$$\int_Y y\left(L_{\mathsf{b}}[v](y) + L_{\mathsf{c}}[v,v](y) + L_{\mathsf{s}}[v,v](y)\right)\ \mathrm{d}y\ =\ 0\ . \tag{33}$$

Taking into account the fact that, by virtue of the Neumann boundary conditions,

$$-\int_\Omega \Delta_1\, w\ \mathrm{d}x = 0\ ,\quad t>0\ ,\quad w\in \mathsf{D}(\Delta_1)\ ,$$

the assertion is a consequence the regularity properties of u. $\square$

5 Global Existence

In this concluding section we derive sufficient conditions ensuring the global-in-time existence of solutions to problem (20). The first result is dedicated to the simplest case of particle size independent diffusion coefficients. This assumption yields *a priori* bounds, pointwise with respect to $x\in\Omega$, as has already been observed in [28]. Subsequently, we consider the general case either if space dimension equals 1 or in the absence of fragmentation.

Throughout, we denote by $u = u(\cdot; u^0)$ the solution to (20) provided by Theorem 4.

Theorem 13 *Let $n \leq 3$ and $p \in (n/2, \infty)$ with $p \geq 1$ and assume that $u^0 \in \mathbb{L}_p^+$. Then u exists globally, that is, $J(u^0) = \mathbb{R}^+$, provided that one of the following conditions is valid:*

(i) for each $T > 0$ there exists $C(T) > 0$ such that

$$\|u(t)\|_{L_\infty(\Omega, L_1(Y))} \leq C(T) \ , \quad t \in J(u^0) \cap [0, T] \ ;$$

(ii) there exists $k_0 > 0$ with

$$K(y, y') \leq k_0 \, (y + y') \ , \quad a.e. \ (y, y') \in Y \times Y \ , \tag{34}$$

and, for each $T > 0$, there exists $C(T) > 0$ such that

$$\|u(t)\|_{L_\infty(\Omega, L_1(Y, y\mathrm{d}y))} \leq C(T) \ , \quad t \in J(u^0) \cap [0, T] \ .$$

Proof. It follows as in Lemma 11 that, for $v \in L_1(Y, (1 + y)\mathrm{d}y)$, one has

$$\begin{aligned} \|L[v]\|_{L_1(Y, \mathrm{d}y)} &\leq c \, (1 + \|v\|_{L_1(Y, \mathrm{d}y)}) \, \|v\|_{L_1(Y, \mathrm{d}y)} \ , \\ \|L[v]\|_{L_1(Y, y\mathrm{d}y)} &\leq c \, (1 + \|v\|_{L_1(Y, \mathrm{d}y)}) \, \|v\|_{L_1(Y, y\mathrm{d}y)} \ , \end{aligned}$$

under the general assumptions $(H_1) - (H_4)$, whereas (34) implies that

$$\|L[v]\|_{L_1(Y, \mathrm{d}y)} \leq c \, (1 + \|v\|_{L_1(Y, y\mathrm{d}y)}) \, \|v\|_{L_1(Y, \mathrm{d}y)} \ .$$

Hence, in both cases (i) and (ii), we see

$$\|L[u(t)]\|_{\mathbb{L}_1} \leq c(T) \, \|u(t)\|_{\mathbb{L}_1} \leq c(T) \, \|u(t)\|_{\mathbb{L}_p} \ , \quad t \in J(u^0) \cap [0, T] \ .$$

Since $n < 4$, we may assume that $n(1 - 1/p)/2 < 1$ by making $p > n/2$ smaller if necessary. Therefore, applying the singular Gronwall inequality (cf. [4, Cor.II.3.3.2]) to the estimate

$$\begin{aligned} \|u(t)\|_{\mathbb{L}_p} &\leq \|u^0\|_{\mathbb{L}_p} + \int_0^t \|U(t - s)\|_{\mathcal{L}(\mathbb{L}_1, \mathbb{L}_p)} \, \|L[u(s)]\|_{\mathbb{L}_1} \, \mathrm{d}s \\ &\leq \|u^0\|_{\mathbb{L}_p} + c(T) \int_0^t (t - s)^{-\frac{n}{2}(1 - \frac{1}{p})} \, \|u(s)\|_{\mathbb{L}_p} \, \mathrm{d}s \ , \end{aligned}$$

we deduce that

$$\|u(t)\|_{\mathbb{L}_p} \leq c(T) \ , \quad t \in J(u^0) \cap [0, T] \ .$$

Recalling Theorem 7, this implies $J(u^0) = \mathbb{R}^+$. $\square$

For diffusion coefficients independent of the particle size y, sufficient conditions for (i) or (ii) are collected in the next corollary. Roughly, solutions exist globally if, for small particles, either collisional breakage is dominated by coagulation, or collision at all is not very frequent. These results are slightly more general than those contained in [28, Cor.5.7].

Corollary 14 *Let $n \le 3$ and $p \in (n/2, \infty)$ with $p \ge 1$ and assume that the diffusion coefficients are independent of $y \in Y$. For $u^0 \in \mathbb{L}_p^+$ suppose that, in addition, one of the following conditions is valid:*

(i) u^0 belongs to $L_\infty(\Omega, L_1(Y,(1+y)\mathrm{d}y))$, and there exists $z_0 \in Y$ such that

$$Q(y,y')\left(\int_0^{y+y'} \beta_{\mathsf{c}}(y+y',y'')\,\mathrm{d}y''-2\right) \le P(y,y')\ , \quad y+y'<z_0\ ; \tag{35}$$

(ii) u^0 belongs to $L_\infty(\Omega, L_1(Y, y\mathrm{d}y))$ and there exists $k_0 > 0$ with

$$K(y,y') \le k_0\,(y+y')\ , \quad a.e.\ (y,y') \in Y \times Y\ .$$

Then u exists globally.

Proof. Define

$$w_k(t,x) := \int_Y y^k\, u(t,y,x)\,\mathrm{d}y\ , \quad (t,x) \in J(u^0) \times \Omega\ , \quad k=0,1\ ,$$

and observe that $\dot{w}_1 - d\Delta_1 w_1 = 0$, due to (33). Hence

$$\|w_1(t)\|_{L_\infty} = \|e^{td\Delta_1} w_1(0)\|_{L_\infty} \le \|w_1(0)\|_{L_\infty} < \infty\ , \quad t \in J(u^0)\ , \tag{36}$$

since $e^{td\Delta_1}$, $t \ge 0$, restricts to a semigroup of contractions on L_∞ (see [21]). By Theorem 13 we may focus on case (i) in the following. Lemma 11 and inequality (35) imply that, for $v \in L_1^+(Y,(1+y)\mathrm{d}y)$,

$$\int_Y L[v](y)\,\mathrm{d}y \le c\,(1+\|v\|_{L_1(Y,y\mathrm{d}y)})\,\|v\|_{L_1(Y,\mathrm{d}y)}$$

as has been observed in [25, Thm.2.9]. Consequently, $w_0(t)$ being non-negative, estimate (36) yields

$$\dot{w}_0(t) - d\Delta_1 w_0(t) = \int_Y L[u(t)]\,\mathrm{d}y \le c\, w_0(t)\ , \quad t \in J(u^0)\ .$$

Therefore,

$$\|w_0(t)\|_{L_\infty} \le e^{ct}\,\|w_0(0)\|_{L_\infty}\ , \quad t \in J(u^0)\ ,$$

which shows that condition (i) is satisfied. □

Now we consider the case that the space dimension equals 1, for which criterion (25) allows to give a much simpler proof for global existence than the ones in

[5, Thm.8.1.] and [28, Thm.5.11.].
For this purpose, let us assume that, similarly as in (35), collisional breakage is dominated by coalescence, i.e., that

$$Q(y,y')\left(\int_0^{y+y'} \beta_{\mathsf{c}}(y+y',y'')\,\mathrm{d}y''-2\right)\le P(y,y')\ ,\quad y+y'\in Y\ , \tag{37}$$

and that scattering is a binary processes, meaning that

$$\beta_{\mathsf{s}}(y,y')=\beta_{\mathsf{s}}(y,y-y')\ ,\quad 0<y-y_0<y'<y_0\ , \tag{38}$$

and

$$\beta_{\mathsf{s}}(y,y')=0\ ,\quad 0<y'<y-y_0<y_0\ . \tag{39}$$

The latter assumption is mandatory since each of the daughter particles y' and $y-y'$ in (38) has to belong to Y. Note that hypothesis (H_4) and (38), (39) imply the identities

$$\int_0^{y_0} \beta_{\mathsf{s}}(y,y')\,\mathrm{d}y'=2\ ,\quad a.e.\ y\in(y_0,2y_0)\ . \tag{40}$$

Referring to Example 1, we observe that (37) is satisfied provided that

$$\frac{-\zeta}{\zeta+1}\,Q(y,y')\le P(y,y')\ ,\quad y+y'<y_0\ ,$$

and (38) is valid if $\nu=0$.

Theorem 15 *Suppose that $n=1$ and let* (37)-(39) *be satisfied. Then u exists globally for each $u^0\in\mathbb{L}_1^+$.*

Proof. Notice that hypotheses $(H_2)-(H_4)$, Lemma 11, (37), and (40) imply that, since $u(t)$ is non-negative,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_Y u(t)\,(1+y)\,\mathrm{d}y &= \int_\Omega\int_Y \Big(L_{\mathsf{b}}[u]+L_{\mathsf{c}}[u,u]+L_{\mathsf{s}}[u,u]\Big)\,(1+y)\,\mathrm{d}y\mathrm{d}x\\ &\le m_\gamma\int_\Omega\int_Y u(t)\,(1+y)\,\mathrm{d}y\mathrm{d}x\end{aligned}$$

for $t\in\dot J(u^0)$. It follows $\|u(t)\|_{\mathbb{L}_1}\le c(T)$ for $t\in J(u^0)\cap[0,T]$, where $T>0$ is arbitrary. The assertion is now a consequence of (25). $\square$

We turn to the main result concerning global existence of solutions. Namely, we prove that solutions exist globally-in-time for small initial values in the absence of fragmentation, but for general diffusion coefficients and for any space dimension. This result is based on (13), (14) and makes use of the quadratic terms L_{c} and L_{s}. More precisely, it relies on the facts that an *a* *priori* estimate in $\mathbb{P}(\mathbb{L}_p)$ is available and that the restriction of $-\mathbb{A}_p$ to $\mathbb{L}_p^\bullet$ has a negative spectral bound.

Theorem 16 *Suppose that there is no fragmentation, i.e.* $\gamma \equiv 0$. *Let (37)-(39) be satisfied and assume that* $p \in (n/2, \infty)$ *and* $p \geq 2$. *Then there exists* $R_0 > 0$ *such that the solution* u *is bounded in* $\mathbb{L}_p$ *whenever* $u^0 \in \mathbb{L}_p^+$ *with* $\|u^0\|_{\mathbb{L}_p} \leq R_0$. *In particular,* $J(u^0) = \mathbb{R}^+$ *in this case.*

Proof. Let $\mathbb{P}$ denote the projection introduced in section 2. Decompose u into $u = w + v$, where

$$w := \mathbb{P}u \in C(J(u^0), L_1^+(Y, (1+y)\mathrm{d}y)) \cap C^1(\dot{J}(u^0), L_1(Y, (1+y)\mathrm{d}y)) \ ,$$

and

$$v := (1-\mathbb{P})u \in C(J(u^0), \mathbb{L}_p) \cap C^1(\dot{J}(u^0), \mathbb{L}_q) \cap C(\dot{J}(u^0), \mathbb{H}^2_{q,\mathcal{B}}) \ , \quad q \in (1, \infty) \ .$$

Taking Proposition 3 into account, we see that (w, v) solves the system

$$\begin{aligned} \dot{w} &= \mathbb{P}\Big(L_{\mathsf{c}}[u,u] + L_{\mathsf{s}}[u,u]\Big) \ , \\ \dot{v} + \mathbb{A}_p^\bullet v &= (1-\mathbb{P})\Big(L_{\mathsf{c}}[u,u] + L_{\mathsf{s}}[u,u]\Big) =: f(w,v) \ , \end{aligned}$$

where $\mathbb{A}_p^\bullet$ is the $\mathbb{L}_p^\bullet$-realization of $\mathbb{A}_p$. As in the proof of Theorem 15 we infer from $(H_3), (H_4)$, Lemma 11, (37), and (40) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_Y w(t)\,(1+y)\ \mathrm{d}y = \frac{1}{|\Omega|} \int_\Omega \int_Y \Big(L_{\mathsf{c}}[u,u] + L_{\mathsf{s}}[u,u]\Big)\,(1+y)\ \mathrm{d}y\mathrm{d}x \leq 0 \ .$$

Because w is independent of $x \in \Omega$, we therefore have

$$\|w(t)\|_{\mathbb{L}_p} \leq \|w(0)\|_{\mathbb{L}_p} \ , \quad t \in J(u^0) \ . \tag{41}$$

Next observe that

$$\begin{aligned} \|f(w,v)\|_{\mathbb{L}_{p/2}^\bullet} &\leq \|1-\mathbb{P}\|_{\mathcal{L}(\mathbb{L}_{p/2}, \mathbb{L}_{p/2}^\bullet)}\, \|L_{\mathsf{c}}[u,u] + L_{\mathsf{s}}[u,u]\|_{\mathbb{L}_{p/2}} \\ &\leq c_0 \left(\|w\|^2_{\mathbb{L}_p} + \|v\|^2_{\mathbb{L}_p^\bullet}\right) \ . \end{aligned}$$

According to Proposition 3, there exist $M \geq 1$ and $\omega_0 > 0$ such that

$$\|U(t)\|_{\mathcal{L}(\mathbb{L}_p^\bullet)} \leq M\, e^{-\omega_0 t} \ , \quad t \geq 0 \ ,$$

and

$$\|U(t)\|_{\mathcal{L}(\mathbb{L}_{p/2}^\bullet, \mathbb{L}_p^\bullet)} \leq M\, e^{-\omega_0 t}\, t^{-n/2p} \ , \quad t > 0 \ .$$

Let Γ denote the gamma function and put

$$\delta := \Big(8\, M\, c_0\, \omega_0^{n/2p-1}\, \Gamma(1-n/2p)\Big)^{-1} \ .$$

Now assume that $\|u^0\|_{\mathbb{L}_p} \le R_0$, where $R_0 := \delta/8M$. Since then $\|v(0)\|_{\mathbb{L}_p^\bullet} \le \delta/4$, it follows that

$$t^* := \sup \left\{ \tau \in J(u^0)\, ;\, \|v(t)\|_{\mathbb{L}_p^\bullet} < \delta\, ,\, 0 \le t < \tau \right\} > 0\ .$$

Consequently, due to $\|w(0)\|_{\mathbb{L}_p} \le \delta$, we have, for $0 < t < t^*$,

$$\begin{aligned}
\|v(t)\|_{\mathbb{L}_p^\bullet} &\le M\, e^{-\omega_0 t}\, \|v(0)\|_{\mathbb{L}_p^\bullet} + \int_0^t \|U(t-s)\|_{\mathcal{L}(\mathbb{L}_{p/2}^\bullet, \mathbb{L}_p^\bullet)}\, \|f(w(s), v(s))\|_{\mathbb{L}_{p/2}^\bullet}\, \mathrm{d}s \\
&\le \delta/4 + M\, c_0 \int_0^t e^{-\omega_0(t-s)}\, (t-s)^{-n/2p} \left(\|w(s)\|_{\mathbb{L}_p}^2 + \|v(s)\|_{\mathbb{L}_p^\bullet}^2 \right) \mathrm{d}s \\
&\le \delta/4 + M\, c_0 \left(\|w(0)\|_{\mathbb{L}_p}^2 + \delta^2 \right) \int_0^t e^{-\omega_0(t-s)}\, (t-s)^{-n/2p}\, \mathrm{d}s \\
&\le \delta/4 + 2\, \delta^2\, M\, c_0\, \omega_0^{n/2p-1}\, \Gamma(1 - n/2p) \\
&\le \delta/2\ .
\end{aligned}$$

From this we readily infer that $t^* = \sup J(u^0)$, that is, $\|v(t)\|_{\mathbb{L}_p^\bullet} \le \delta$ for $t \in J(u^0)$. Estimate (41) thus yields that u is bounded in $\mathbb{L}_p$. Hence, u exists for all time according to Theorem 7. $\square$

Remarks 17 **(a)** *Since $\mathbb{L}_p$ embeds in $\mathbb{L}_1$, this theorem implies, in particular, that the total number of particles is bounded above. (Recall that from Theorem 12 we know that the total mass is preserved.)*

(b) *Note that the solution u exists globally also for $\gamma \not\equiv 0$ provided that there is no collision of particles, i.e. $K \equiv 0$, since in this case the evolution equation is linear.*

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